SIMPLIFIED DERIVATION OF THE PROPERTIES OF ELEMENTARY TRANSCENDENTALS

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The purpose of the present paper is to give a derivation of the properties of the elementary transcendentals from a more unified point of view than is customary. It will be assumed that the general properties of analytic functions are established. In particular we shall use the theorem that the real (or imaginary) part of an analytic function cannot assume its maximum value in the inside of a closed domain in which it is regular. Similarly, use will be made of Weierstrass's theorem about a set of uniformly converging analytic functions, of Liouville's and Rolle's theorems.§ All of these theorems can be established without the use of the properties of the transcendental functions, using only elementary calculations and the concept of convergence. The present paper shows how the properties of the elementary transcendentals can then be derived from the aforementioned theorems by means of the methods of the theory of complex functions. From the point of view of this theory, this appears to be more appropriate than taking over the results concerning the properties of these functions, at least for real arguments, from other parts of analysis. An outline of our procedure follows.

As a first step, a special class of meromorphic functions, the so-called R functions, will be introduced and a number of properties of these functions, including two expansion theorems, established (Section 1). Some of these properties will not be used for the derivation of the properties of elementary transcendentals; the numbers of the corresponding theorems have a star attached to them.

The tangent then will be defined in Section 2 as the simplest periodic R function. The partial fraction and infinite product expansion of the tangent is a special case of the expansion theorems mentioned before while the addition theorem for this function will follow from its periodic nature. It is easy to obtain then the similar expansions for the other trigonometric and the exponential functions.

The writer wishes to admit at the outset that his definition of the exponential function is less direct than that of Caratheodory,† for instance. On the other hand, the derivation of the various expansion theorems which will be given below is less artificial and more general than the customary derivations.

§ *Cf. e.g.* A. Hurwitz and R. Courant, *Allgemeine Funktionentheorie etc.* Verlag Julius Springer, Berlin 1922, III 2, page 296; I 3, p. 63 and I 3, p. 56.

† C. Caratheodory, *Funktionentheorie*, Verlag Birkhauser, Basel 1950, Sec. 234, p. 233.

1. Properties of the R functions. R functions are defined ‡ as meromorphic functions of a complex variable z, the imaginary part of which is non-negative in the upper and non-positive in the lower half-plane. For the sake of the reader's convenience, some rather obvious properties of such functions are repeated here from an earlier article of the writer.‡ A few of these properties had been given previously by Fränz.‡

THEOREM 1.1. *If a set of R functions converges to a meromorphic function, this function itself is an R function.*

THEOREM 1.2. *An R function is real on the real axis and assumes real values only on the real axis.*

THEOREM 1.3. *The derivative of an R function is positive at every regular point on the real axis unless R is a constant.*

THEOREM 1.4. *All poles of an R function are on the real axis, they are simple and their residues negative.* The poles will be denoted henceforth by Z_μ, the corresponding residues by $-\gamma_\mu^2$; both the Z and γ are then real. A typical R function for real values of its argument is given in Figure 1.

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‡ As far as the writer could determine, R functions appear in the mathematical literature first (though not under this name) in K. Loewner's article *Ueber monotone Matrixfunktionen*, Math. Zeits. 38, 177 (1933). In this article Loewner considers functions $R(m)$ of real symmetric or hermitean matrices m which satisfy the condition that $R(m_1)-R(m_2)$ be positive definite if m_1-m_2 is positive definite and if the characteristic values of m_1 and m_2 are in an interval (λ, λ'). He shows that the $R(m)$ are R functions which are regular in the (λ, λ') interval. The writer is much indebted for this quotation to S. Sherman. Certain generalizations of R functions also play an important role in the theory of moments. *Cf. e.g.* J. A. Shohat and J. D. Tamarkin, The Problem of Moments, Publication of the American Mathematical Society, New York 1943, particularly Chapters I and II. The R functions have also played a substantial if not always obvious role in electrical engineering, in particular in the theory of the reactance of circuits, ever since the pioneering investigations of O. J. Zobel, G. A. Campbell and R. M. Foster (Bell System Technical Journal, Volumes 1, 2 and 3, 1922–24). *Cf.* also W. Cauer, Preuss. Akad. Ber. 1931, p. 30. A more modern review of the theory has been given recently by H. W. Bode, Network Analysis and Feedback Amplifier Design (Van Nostrand, 1945). However, most of this work deals with rational R functions for which most of the contents of the present article are obvious. An exception is the article of K. Fränz, Elektrische Nachrichten Technik, 21, 8 (1944) who already considers transcendental R functions. The writer is much indebted for this quotation to Dr. T. Teichmann. For odd R functions, the expansion theorems 1.10 and 1.12 also follow from the work of P. I. Richards, Duke Math. Journ., 14, 777, 1947, which is based on W. Cauer's article in Bull. Amer. Math. Soc., 38, 713, 1932. This article anticipates, for odd R functions, many of the results of our Section 1. The connection between his *iPRF* function $Z(s)$ and the R functions is $Z(s)=-iR(is)$. *Cf.* also Richards' summary article, Quarterly Journ. of Math., 6, 21, 1948. The writer's interest in these functions was raised by his work on the quantum mechanical theory of reaction and scattering processes. *Cf.* in this connection E. P. Wigner and L. Eisenbud, Physical Rev. 72, 29 (1947) and the summary article Amer. Journ. of Physics, 17, 99 (1949). The first general investigation of the mathematical properties of these functions is given in Ann. of Math. 53, 36 (1951). It was not entirely possible to eliminate all repetition between this last article and the present one.

THEOREM 1.5. *A linear combination of R functions with positive coefficients is an R function.*

THEOREM 1.6. *The linear function $\alpha z+\beta$ is an R function if α is positive and β real.*

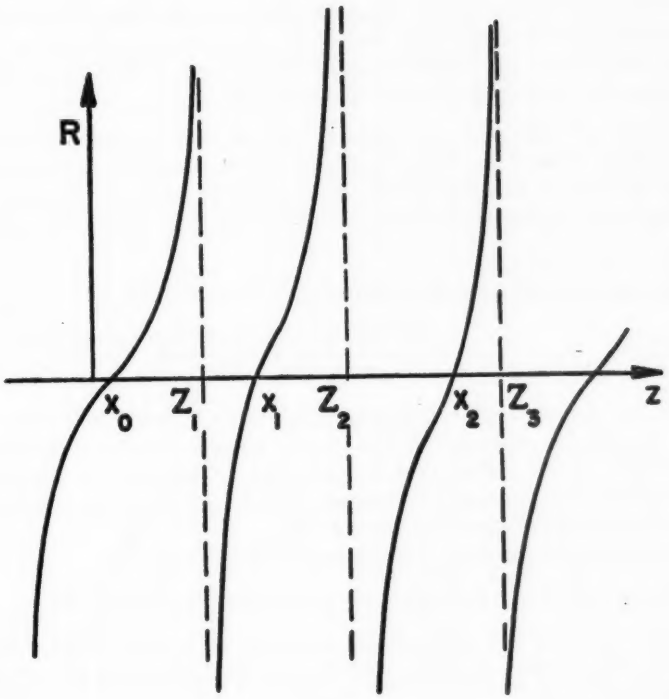

FIG. 1

THEOREM 1.7. *A linear fractional transform of an R function*

$$S = \frac{a_1R + b_1}{a_2R + b_2} \tag{1}$$

with real coefficients and positive determinant $a_1b_2-b_1a_2$ is again an R function.

THEOREM 1.8. *A real rational function is an R function if*

a) *the degree of its numerator does not exceed that of the denominator by more than one,*

b) *its value remains below a finite number as $z\to-\infty$,*

c) *its poles are real and their residues negative.*

The rational function in question can be written in the form

$$R(z) = \alpha z + \beta + \sum\left(\frac{\gamma_\mu^2}{Z_\mu - z} - \frac{\gamma_\mu^2}{Z_\mu}\right) \tag{2}$$

the number of terms in the sum being the degree of the denominator. Because of the second condition, $\alpha \geqq 0$, because of the last condition Z_μ and γ_μ are real; β is real also. Hence, by Theorem 1.6 and Theorem 1.7, all terms of (2) are R functions and it is an R function by Theorem 1.5.

THEOREM 1.9. *The series* (2), *finite or infinite, with positive or vanishing* α, *real* β, γ *and* Z *is an* R *function if the* Z_μ *have no finite condensation point and if the series on the right converges at least for one value of* $z \neq 0$. *In this case, it converges absolutely and uniformly in every closed domain of* z *which excludes the poles* Z.

In fact, the general term of the series in (2) has the form

$$\frac{\gamma^2}{Z - z} - \frac{\gamma^2}{Z} = \left(\frac{\gamma}{Z}\right)^2 z\left(1 + \frac{z}{Z - z}\right)$$

which shows: if the series converges for a $z \neq 0$, the series with the positive general term $(\gamma/Z)^2$ converges too, and so the series in (2) converges absolutely and uniformly in every closed domain excluding the poles Z, and represents, by Weierstrass' theorem, an analytic function. This function is clearly meromorphic and an R function by Theorems 1.8 and 1.1.

We now come to the expansion theorems for R functions.

THEOREM 1.10. *Every* R *function can be expanded in the form* (2).

In order to prove this without the use of the properties of the elementary transcendentals, we first establish the following Lemma due to M. Schiffer and V. Bargmann.§

LEMMA 1.10a. *Let* $R(z)$ *be an* R *function,* Z_1 *one of its poles and* $\gamma_1^2/(Z_1 - z)$ *its meromorphic part at* Z_1. *Then*

$$R_1(z) = R(z) - \gamma_1^2/(Z_1 - z) \tag{3}$$

is also an R *function.*

The proof of this theorem, due to Schiffer and Bargmann, first observes that R_1 is clearly meromorphic. If its imaginary part assumed a negative value $-\rho$ at a point z' of the upper half-plane, one could draw a circle about Z_1 with a radius larger than $2\gamma_1^2/\rho$ and large enough to contain z'. The imaginary part of R_1 is larger than $-\frac{1}{2}\rho$ along the part of this circle which lies in the upper half-

§ Cf. E. P. Wigner, Ann. of Math. 53, 36 (1951). Only the last part of the present proof, beginning with lemma 1.10b, is new.

plane because the imaginary part of R is positive and the absolute value of $\gamma_1^2/(Z_1-z)$ is smaller than $\frac{1}{2}\rho$ along this semicircle. Furthermore, R_1 is real at every point of regularity of the real axis, including Z_1. Finally, if Z_μ is a pole of R within the circle, one can draw a sufficiently small semicircle about Z_μ into the upper half-plane so that the imaginary part of R_1 becomes positive thereon. In order to demonstrate this, one observes that, by Theorem 1.4, Z_μ is a simple pole of R_1. Hence there is a neighborhood $|\zeta|<\epsilon$ of Z_μ with $\zeta=\zeta_1+i\zeta_2=Z_\mu-z$ where $R_{1\mu}=R_1-\gamma_\mu^2/\zeta$ is regular. Furthermore $R_{1\mu}$ remains real on the real axis so that, by Rolle's theorem, the imaginary part of $R_{1\mu}$ for $|\zeta|<\epsilon$, $\zeta_2>0$ is larger than $M\zeta_2$ where M is the minimum of the derivative of $R_{1\mu}$ in the direction of the imaginary axis. Hence the imaginary part of R_1 is, in the same region, larger than $(M+\gamma_\mu^2/|\zeta|^2)\zeta_2$. The bracket of this expression can however be made positive by choosing $|\zeta|^2$ small enough. Thus z' can be surrounded by a continuous path (*cf.* Figure 2) containing no singularities of R_1, on which the imaginary part of R_1 is everywhere larger than $-\frac{1}{2}\rho$. Hence the imaginary part of R_1 cannot be negative in the upper half-plane and a similar argument shows that it cannot be positive in the lower half-plane. Thus R_1 is an R function.

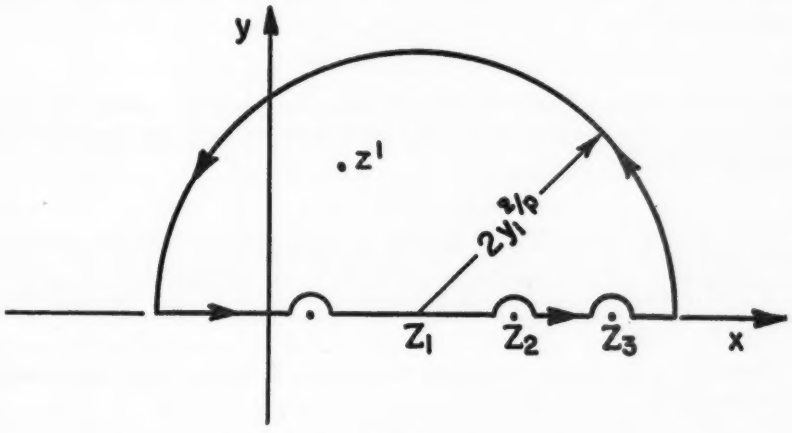

FIG. 2

We can subtract in a similar way any finite number of singularities from R, and the resulting function

$$R_n = R - \sum[\gamma_\mu^2/(Z_\mu - z) - \gamma_\mu^2/Z_\mu] \tag{4}$$

will remain an R function. Hence, its derivative at a real point of regularity z_0 will be positive by Theorem 1.3:

$$\sum\gamma_\mu^2/(Z_\mu - z_0)^2 < R'(z_0). \tag{5}$$

It follows that the whole series $\gamma_\mu^2/(Z_\mu - z_0)^2$ converges if z_0 is real. However, since $|\gamma_\mu^2/(Z_\mu - x - iy)^2| < \gamma_\mu^2/(Z_\mu - x)^2$, the series (5) converges absolutely also for complex z_0. The same is true of the series

$$\left|\frac{\gamma_\mu^2}{Z_\mu - z_0} - \frac{\gamma_\mu^2}{Z_\mu}\right| = |z_0| \sum \left|\frac{\gamma_\mu^2}{(Z_\mu - z_0)Z_\mu}\right|. \tag{6}$$

Except for a finite number of terms, its terms are smaller than $2|z_0|$ times the corresponding terms of (5). Hence, the conditions of Theorem 1.9 are fulfilled and R_n converges, as we subtract all the poles, by the second part of Theorem 1.9, to an analytic function $I(z)$. This $I(z)$ can have no finite singularity. As a limiting case of R functions, $I(z)$ is itself an R function. In order to prove Theorem 1.10, it remains to be shown only that such a function must be linear. This follows from Lemma 1.10b:

LEMMA 1.10b: *An entire R function $I(z)$ is a linear function $\alpha z + \beta$, with real β, $\alpha \geqq 0$.*

Because of Theorem 1.3, unless $I(z)$ is a constant, it is for real z a monotonically increasing function. Let us denote its value at $z = 0$ with β and consider the function

$$J(z) = 1/(\beta - I(z)). \tag{7}$$

This is, by Theorem 1.7, an R function, negative for positive z, positive for negative z. It has only one singularity, at $z = 0$. Let us subtract this singularity by Theorem 1.10a. The result

$$J_1(z) = J(z) + \gamma^2/z \tag{8}$$

is again an entire R function. It must be a constant unless it is monotonically increasing. Since $J(z)$ is negative for positive z, and conversely, $J_1(\infty) \leqq 0$ and $J_1(-\infty) \geqq 0$. Hence $J_1(z) = 0$ and $I(z) = \beta + z/\gamma^2$ which proves Theorem 1.10b and hence, Theorem 1.10.

THEOREM 1.11. *Given a set of real numbers x_μ, Z_μ without finite condensation points for which*

$$\cdots < Z_{-2} < x_{-2} < Z_{-1} < x_{-1} < Z_0 < x_0 < Z_1 < x_1 < Z_2 < x_2 < \cdots \tag{9}$$

and $Z_0 \leqq 0 < Z_1$, $c > 0$, the products (finite or infinite) in

$$R(z) = c\frac{x_0 - z}{Z_0 - z}\prod_\nu \frac{1 - z/x_\nu}{1 - z/Z_\nu}\prod_\nu \frac{1 - z/x_{-\nu}}{1 - z/Z_{-\nu}} \tag{9a}$$

converge absolutely and uniformly in every finite domain to an R function.

The ν in both products of (9a), as in all formulae below, runs over all positive integers. If the set (9) breaks off at a finite Z_l, the last factor in the first $\prod$ is $(1 - z/Z_l)^{-1}$; if the set starts with a finite x_s the last factor in the second

$\prod$ is $z-x_s$. The last two rules can be replaced by the convention of adding an $x_l=\infty$ to the set of x in the former and a $Z_s=-\infty$ to the set of Z in the latter case. Hence (9) always begins with a Z and ends with an x if it has a finite beginning or end.

To show Theorem 1.11 it will be necessary to establish the well-known theorem about the convergence of infinite products without the use of the exponential function.

LEMMA 1.11a. *The absolute convergence of* $\prod(1+a_\nu)$ *follows from the convergence of* $\sum|a_\nu|$.

Indeed, it can be shown that any partial product $\prod'(1+a_\nu)$ is closer to 1 than the corresponding sum $\sum|2a_\nu|$ as long as this sum does not exceed 1. We shall prove

$$|\prod'(1+a_\nu)-1|<2\sum'|a_\nu| \tag{10}$$

by induction. It is correct if the partial product contains only one factor. Let us assume that (10) is correct if the partial product contains only $n-1$ factors. It then follows that

$$\begin{aligned}|\prod'(1+a_\nu)(1+a_n)-1| &\leqq |\prod'(1+a_\nu)-1|+|a_n||\prod'(1+a_\nu)| \\ &<2\sum'|a_\nu|+|a_n|(1+2\sum'|a_\nu|) \\ &\leqq 2(|a_n|+\sum'|a_\nu|).\end{aligned} \tag{11}$$

The last part holds because $\sum'|a_\nu|\leqq\frac{1}{2}$ by assumption. This proves the inequality (10) for n factors and hence in general. The lemma 1.11a is an immediate consequence of (10).

Given now any closed domain, there is only a finite number of Z_μ the absolute value of which is smaller than the maximum value of $2|z|$ in the domain. Apart from the corresponding factors in the products in (9a), $|1-z/Z_\mu|>\frac{1}{2}$ holds for all z of the domain. The product of these factors will converge absolutely and uniformly by Lemma 1.11a if the series

$$\sum|a_\mu|=\sum\left|\frac{1-z/x_\mu}{1-z/Z_\mu}-1\right|=|z|\sum\left|\frac{Z_\mu^{-1}-x_\mu^{-1}}{1-z/Z_\mu}\right|\leqq 2|z|\sum|1/Z_\mu-1/x_\mu| \tag{12}$$

converges. The last series is, however, because of (9), smaller than $2|z/Z_1|$ for the first product and smaller than $2|z/x_{-1}|$ for the second product in (9a). Thus, the product of the factors considered represents, by Weierstrass' theorem, an analytic function in the domain considered and the right side of (9a) is a meromorphic function.

In order to prove that (9a) is an R function, it is, because of Theorem 1.1, necessary only to prove that every finite partial product of (9a) is an R function. These partial products are real rational functions and the degree of their numerators is equal to that of the denominator, unless the series (9) has a beginning

or end, in which case the degrees differ by ± 1. Second, all factors of (9a) tend to 1 as z goes to $\pm\infty$ save the exceptional factors $(1-z/Z_l)^{-1}$ and $z-x_s$, if such are present. At any rate $R(z)$ is bounded from above as $z\to-\infty$. Finally for $\mu\neq 0$, every factor is positive at $z=Z_\mu$, except

$$(1-z/x_\mu)/(1-z/Z_\mu)=Z_\mu(1-z/x_\mu)/(Z_\mu-z)$$

and the numerator of this expression is also positive at $z=Z_\mu$ both for $\mu<0$ and for $\mu>0$. Hence the residues of all poles Z_μ with $\mu\neq 0$ are negative and one can easily see that this holds also for the pole Z_0. It thus follows from Theorem 1.8 that the finite partial products of (9a) are R functions and from Theorems 1.11a and 1.1 that (9a) is also an R function.

THEOREM 1.12. *Every R function $R(z)$ can be expanded§ into a product of the form* (9a).

Because of Theorem 1.3, the poles Z_μ and zeros x_μ satisfy the conditions (9) whence (9a) represents an R function with the same poles and zeros as $R(z)$. Hence Theorem 1.12 will be proved if it can be shown that the ratio $r(z)$ of two R functions with the same poles and zeros is a constant. Since both zeros and poles of an R function must be, by Theorems 1.3 and 1.4, simple, $r(z)$ is an entire function. It cannot assume a negative real value on the real axis since both R functions are negative in the same (Z_μ, x_μ) intervals, positive elsewhere. It cannot assume a negative real value on the upper half-plane either since the imaginary parts of both R functions are positive there. For the same reason r cannot be negative on the lower half-plane. However, *an entire function $r(z)$ which does not assume any value of a finite interval (r_1, r_2) is necessarily a constant.* One sees this most easily by considering $\rho=((r-r_1)/(r-r_2))^{1/2}$. Since the radicand of this expression assumes negative real values only if r is in the interval (r_1, r_2), and since $r(z)$ does not assume values in this interval, ρ is a single-valued function which does not assume any purely imaginary value. Its real part is either positive for every z, or negative for every z. Application of Liouville's theorem to $1/(\rho+i)$ in the former and $1/(\rho-i)$ in the latter case shows that it is a constant. This then proves Theorem 1.12.

This result about the relation of two R functions which have the same poles and zeros can be generalized to the case when two other real values, a and $b>a$, are assumed by both R functions at the same two sets of values of z. By applying the above result to $(R-a)/(b-R)$ one finds:

THEOREM 1.13.* *If two R functions assume the real value a at the same values of z, and if this is true for another real number b, the two R functions can be obtained from each other by a fractional linear transformation which leaves a and b invariant.*

§ Under certain more restrictive assumptions, this theorem has been shown already by K. Fränz, *l.c.* footnote p. 670.

We now come to a couple of further theorems which are not necessary for the development of Section 2. They give the connection between differential equations and R functions. Let us consider the function $\phi(\xi, z)$ of the complex variable z and real variable ξ which obeys the differential equation

$$\frac{d^2\phi}{d\xi^2} = U(\xi)\frac{d\phi}{d\xi} + V(\xi)\phi - z\phi \tag{13}$$

in which U, V are real continuous functions of ξ in the interval $(0, \xi_0)$. Given z-independent initial conditions at $\xi=0$, it is well known that† ϕ and $d\phi/d\xi$, as functions of z, are entire functions.

THEOREM 1.14.* *The solution $R(\xi, z)$ of Riccati's equation*

$$\frac{dR(\xi, z)}{d\xi} = R^2 + UR - V + z \tag{14}$$

with the initial condition $R(0, z) = R_0$ and real R_0 is, for every value ξ of the interval $(0, \xi_0)$, an R function of z.

One obtains (14) from (13) by the substitution $R = -(d\phi/d\xi)/\phi$. As quotient of two entire functions, R is meromorphic. Let us separate $R = R_r + iR_s$ into real and imaginary parts. The equation for R_s is

$$dR_s/d\xi = 2R_rR_s + UR_s + y \tag{14a}$$

where y is the imaginary part of $z = x+iy$. It is necessary to show now that $R_s(\xi)$ has the same sign as y.

Because of $R_s(0, z) = 0$, this is evident for sufficiently small values of ξ. Furthermore, if y is positive, R_s has a positive slope at a point where $R_s = 0$. Hence, as ξ increases, R_s cannot become negative as long as R obeys the differential equation (14). Under the same condition, R_s cannot become positive if y is negative.

Naturally, R can have singularities and will not obey (14) at these. In order to prove that R_s remains positive for positive y as ξ passes through a singularity ξ_0, we note that $Q = -1/R$ vanishes at, and is regular in the neighborhood of, ξ_0. Hence, in the neighborhood of ξ_0, the imaginary part Q_s of $Q = Q_r + iQ_s$ vanishes more strongly than its derivative. It follows that we can set $Q_s = 0$ when calculating $dQ_s/d\xi$ in the neighborhood of ξ_0. The differential equation for Q_s then shows that $dQ_s/d\xi$ has, in the neighborhood of ξ_0, the same sign as yQ_r^2, *i.e.* as y. According to the preceding paragraph, this is true also for $Q_s = R_s|Q|^2$ as long as $\xi < \xi_0$ so that Q_s, and hence also R_s, retain their sign as ξ passes

† *Cf. e.g.* E. L. Ince, Ordinary Differential Equations. Dover Publications, New York, 1944. Section 3.31, p. 72.

through ξ_0. This then proves‡ Theorem 1.14.

Theorem 1.14 permits the definition of a wide variety of R functions and, of course, the R functions derive their significance from Theorems 1.14 and 1.15. However, the latter will not be proved here.§

In order to formulate Theorem 1.15, we consider an n dimensional real euclidean space E_n with rectangular coordinates $\xi_1, \xi_2, \cdots, \xi_n$. There are two regular closed surfaces S_1 and S in E_n, the former being entirely inside S. Given further a self-adjoint real operator§§ H defined for functions within S and for which

$$Hf = -\Delta f = -(\partial^2/\partial\xi_1^2 + \partial^2/\partial\xi_2^2 + \cdots + \partial^2/\partial\xi_n^2)f \tag{15}$$

holds if f vanishes within S_1 and belongs to the definition domain of the Laplace operator Δ. This operator is defined to be self-adjoint also and applicable to all twice continuously differentiable functions in S which have zero normal derivatives thereon. Let us consider then a solution of the equation

$$H\phi(\xi, z) = z\phi(\xi, z) \tag{16}$$

such that ϕ is, on the surface S, real and independent of z

$$\phi(\xi, z) = \psi(\xi) \quad \text{on } S. \tag{16a}$$

THEOREM 1.15.* *Under the conditions of the preceding paragraph the surface integral*

$$-\int \psi(\xi)\, \mathrm{grad}_n\, \phi(\xi, z)dS = R(z) \tag{17}$$

extended over the surface S is an R function of z; the grad_n *denotes the component of* grad ϕ *that is normal to the surface S.*

For the sake of completeness, three further and rather obvious criteria will be given here which suffice, in conjunction with Theorem 1.14, to prove that the logarithmic derivative of the Bessel functions and of a few other functions, including the Γ function, are R functions and hence permit partial fraction and product representations of the form (2) and (9a).

THEOREM 1.16.* *$aR(bz+c)$ is an R function of z if a, b, c are real and ab positive, and if $R(z)$ is an R function.*

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‡ This theorem, although in a different form, has been known for a long time. *Cf. e.g.* L. Bieberbach, *Theorie der Differentialgleichungen*, Julius Springer, Berlin 1923, Section 2, Chapter III, p. 161.

§ For a plausibility argument cf. E. P. Wigner and L. Eisenbud, reference 1.

§§ For the definition of self-adjoint operators *cf. e.g.* B.v.Sz. Nagy, *Spektraldarstellung Linearer Transformationen des Hilbertschen Raumes*. Vol. 5 of *Ergebnisse der Mathematik und Grenzgebiete*. Springer Verlag, Berlin 1942 Section V4, p. 34.

THEOREM 1.17.* *Let $R_+(z)$ be an R function all the poles of which are positive, then*

$$R_0(z) = \frac{1}{z}\left[R_+(z^2) - R_+(0)\right] \tag{18}$$

is an odd R function. If $R_+(z) > 0$ for all negative z

$$R_0(z) = z\,R_+(z^2) \tag{18a}$$

is also an odd R function.

The first part follows from the expansion (2) for R_+ in which Z_μ can be replaced by $\zeta_\mu^2 = Z_\mu$ with positive ζ_μ. The identity

$$\frac{1}{z}\left(\frac{1}{\zeta_\mu^2 - z^2} - \frac{1}{\zeta_\mu^2}\right) = \frac{1}{2\zeta_\mu^2}\left(\frac{1}{\zeta_\mu - z} + \frac{1}{-\zeta_\mu - z}\right)$$

together with Theorem 1.9 then directly gives (18). In the second case, $-1/R_+(z)$ satisfies the conditions of the first case for R_+. Hence $-(1/z)(1/R_+(z^2) - 1/R_+(0))$ is an R function and remains one if $-R_+(0)/z$ is added to it. Hence (18a) is also an R function.

THEOREM 1.18.* *Conversely, if $R_0(z)$ is an odd R function which is regular at $z=0$, then $\sqrt{z}R_0(\sqrt{z})$ is also an R function of z.*

2. The elementary transcendentals. Let us consider the class t of R functions which (a) are periodic with period 1, (b) have only one singularity for $-\frac{1}{2} < z \leqq \frac{1}{2}$, (c) converge to i as z goes to $i\infty$. The existence of R functions satisfying these conditions will be apparent when the expansion (2) will be found for them. We now proceed to find this expansion.

Let us denote the position of the singularity in the $-\frac{1}{2} < z \leqq \frac{1}{2}$ interval by $\frac{1}{2} - x_0$, its residue by $-c < 0$. Because of the periodicity condition, there will be similar singularities at all points $n - \frac{1}{2} - x_0$, the n being any integer, and only at these points. Hence, if R functions of the class t exist, their expansion by (2) must be of the form

$$t(z) = \alpha z + \beta + c\sum\left(\frac{1}{n - \frac{1}{2} - x_0 - z} - \frac{1}{n - \frac{1}{2} - x_0}\right), \tag{19}$$

where n assumes every integral value from $-\infty$ to ∞.

It is worth while to remark that because of the absolute convergence of the series (19) one can unite the terms $\nu = n$ and $\nu = 1 - n$; ν then will run over all positive integers. Then

$$t(z) = \alpha z + \beta + \sum\left(\frac{c}{\nu - \frac{1}{2} - x_0 - z} - \frac{c}{\nu - \frac{1}{2} + x_0 + z}\right)$$

(19a)
$$-\sum\left(\frac{c}{\nu - \frac{1}{2} - x_0} - \frac{c}{\nu - \frac{1}{2} + x_0}\right)$$

$$= \alpha z + \beta' + \sum\left(\frac{c}{\nu - \frac{1}{2} - x_0 - z} - \frac{c}{\nu - \frac{1}{2} + x_0 + z}\right),$$

since the sums of both brackets of the first equation converge absolutely and the second of these is independent of z.

In order to impose the boundary condition $t(i\infty) = i$, we calculate the sum in (19a) for $z = x + iy$

$$\sum \frac{c}{\nu + \frac{1}{2} - x_0 - z} - \frac{c}{\nu + \frac{1}{2} + x_0 + z}$$

(19b)
$$= \sum \frac{2c(x_0 + x + iy)}{(\nu + \frac{1}{2})^2 - (x_0 + x + iy)^2}$$

$$= \frac{2ic}{y} \sum \frac{1 - i(x_0 + x)/y}{((\nu + \frac{1}{2})/y)^2 + (1 - i(x_0 + x)/y)^2}.$$

The factor by which the νth term of (19b) differs from $[1+((\nu+\frac{1}{2})/y)^2]^{-1}$ converges to 1 as y goes to infinity. Hence the sum (19b) converges to the sum of these expressions and, by the definition of the integral, (19b) converges to

(19c)
$$\frac{2ic}{y} \sum \frac{1}{1 + (\nu + \frac{1}{2})^2/y^2} \to 2ic \int_0^\infty \frac{d\xi}{1 + \xi^2} = i\pi c$$

if we denote the last integral by $\frac{1}{2}\pi$.

Since t converges to a fixed value i as $z \to i\infty$ and since this is true also of the last term in (19a), $\alpha z + \beta'$ must also converge to a fixed number. This is possible only if $\alpha = 0$ and since β' is real, it also must vanish. It then follows that $\pi c = 1$.

THEOREM 2.1. *The R functions of the class t have the form $t(x_0+z)$ with real x_0 where*

$$t(z) = \frac{1}{\pi} \sum\left(\frac{1}{\nu - \frac{1}{2} - z} - \frac{1}{\nu - \frac{1}{2} + z}\right) = \frac{1}{\pi} \sum\left(\frac{1}{n - \frac{1}{2} - z} - \frac{1}{n - \frac{1}{2}}\right) \tag{20}$$

where ν runs over all positive integers, n over all integers.

It is easy to convince oneself that the t given by (20) fulfills the three conditions of the definition of the class. This is evident for (b) and the above calculation verifies (c). To verify (a) let N be so large that the first N terms of the series (20) represent both $t(z)$ and $t(z+1)$ with a smaller error than ϵ, then (20) shows that $t(z) - t(z+1)$ is within 2ϵ equal to

$$\frac{1}{N-\frac{1}{2}-z}+\frac{1}{N+3/2+z}$$

and this goes to 0 as N increases.

The function $t(z)$ has singularities at $\pm\frac{1}{2}$ and assumes in the interval $(-\frac{1}{2}, \frac{1}{2})$ every real value. Let us consider therefore the R functions

$$\frac{\alpha t(z)+\beta}{-\beta t(z)+\alpha} \tag{21}$$

with real α, β (*cf.* Theorem 1.7). The function (21) is clearly periodic, has only one singularity for $-\frac{1}{2}<z\leqq\frac{1}{2}$, namely at the point where $t(z)=\alpha/\beta$, and one can verify directly that its value is i for $z=i\infty$. It therefore is in the class t and is equal to $t(x_0+z)$ with a definite x_0. In order to determine this x_0, one may insert $z=0$ in (21), because of $t(0)=0$, its value then is $\beta/\alpha=t(x_0)$. Introducing this into (21) gives the addition theorem

$$t(x_0+z)=\frac{t(z)+t(x_0)}{1-t(x_0)t(z)}. \tag{21a}$$

The above derivation proves (21a) only for $-\frac{1}{2}<x_0\leqq\frac{1}{2}$. However, since both sides of (21a) are analytic functions of x_0, it must hold also in general.

The function $t(z)$ is odd, so vanishes at $z=0$. In order to calculate the value of its derivative t' at $z=0$, we note from (21a) with $x_0=1/2$ that

$$t(\tfrac{1}{2}+z)=-1/t(z). \tag{22}$$

The residue of the left side of (22) at $z=0$ is $-1/\pi$, hence the derivative of $t(z)$ at $z=0$ is

$$t'(0)=\pi. \tag{22a}$$

From the odd and real nature of t it follows that it is imaginary on the imaginary axis and, because of (22), it is also imaginary if z has the form $\frac{1}{2}+iy$.

The poles of $-1/t(z)$ are all the integers. From this and its odd nature follows the expansion

$$-\frac{1}{t(z)}=-\frac{1}{\pi z}+\frac{1}{\pi}\sum\left(\frac{1}{\nu-z}-\frac{1}{\nu+z}\right). \tag{23}$$

Let us calculate, finally, the addition formula for an arbitrary fractional linear transform of t

$$e(z)=\frac{a_1t(z)+b_1}{a_2t(z)+b_2}.$$

For $e(x_0+z)$ one obtains a fractional linear function of $e(z)$ with the matrix

$$\begin{Vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{Vmatrix} \begin{Vmatrix} 1 & t(x_0) \\ -t(x_0) & 1 \end{Vmatrix} \begin{Vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{Vmatrix}^{-1}.$$

This will be a diagonal matrix if $a_1=-a_2=i$; $b_1=b_2=1$. Hence the addition theorem for

$$e(z) = \frac{it(z)+1}{-it(z)+1} \tag{24}$$

has the form $e(x_0+z)=f(x_0)e(z)$. Writing $z=0$ herein one finds $f(x_0)=e(x_0)$ and

$$e(z+z') = e(z)e(z'). \tag{24a}$$

Evidently, $e(z)$ is periodic with period 1. From (24) one finds $e(0)=1$, $e(i\infty)=0$, $e(-i\infty)=\infty$, that $|e(z)|=1$ for real z and from the remark after (22a) that it is real for imaginary z: it assumes every positive value as z drops from $i\infty$ to $-i\infty$.

The value of the derivative e' of e at $z=0$ can be obtained from (24) and (22a): $e'(0)=2it'(0)=2\pi i$. The differential equation

$$e'(z) = 2\pi i e(z) \tag{25}$$

then easily follows from (24a). It shows that $e(z)$ can have no poles and is an entire function. The various formulae for the trigonometric and hyperbolic functions follow either from (21a) or (24a) by means of

$$\begin{aligned} t(z) &= \tan \pi z, \\ e(z) &= e^{2\pi i z}. \end{aligned} \tag{26}$$

The infinite product representation for the tangent is a direct consequence of Theorem 1.12:

$$\begin{aligned} t(z) &= \frac{\pi}{2}\,\frac{z}{z+\frac{1}{2}} \prod \frac{1-z/\nu}{1-z/(\nu-\frac{1}{2})} \prod \frac{1+z/\nu}{1+z/(\nu+\frac{1}{2})} \\ &= \frac{\cdots(1-z/2)(1-z)z(1+z)(1+z/2)\cdots}{\cdots(1-2z/3)(1-2z)(1+2z)(1+2z/3)(1+2z/5)\cdots}. \end{aligned} \tag{27}$$

Calculating $t'(z)/t(z)$, one obtains the partial fraction representation of $s(2z)$ and, hence, of $s(z)$ and $c(z)$. The most obvious way to obtain the infinite product representation for $c(z)=\cos \pi z$ is to show that it satisfies the differential equation $c'(z)/c(z)=-\pi t(z)$. The infinite product for the sine can be deduced either in the same fashion or by multiplying the infinite products for $c(z)$ and $t(z)$.

It is worth while yet to point out that every complex number $x+iy$ can be expressed in the form $re(\phi)$ with real r and ϕ. Set $r=(x^2+y^2)^{1/2}$ and $t(\phi)=(r-x)/y=y/(r+x)$; there is one and only one ϕ between $-\frac{1}{2}$ and $\frac{1}{2}$ which satisfies this last equation. This then shows that $\pi\phi$ is the angle in the Gauss plane. The remark after (24a) shows, furthermore, that there is an imaginary

number $i\rho$ such that $r=e(i\rho)$ if r is positive. Hence, unless $x+iy=0$, there is a complex $\phi+i\rho$ such that $e(\phi+i\rho)=x+iy$. This defines the logarithm; its properties, including its derivative, follow from the properties of the exponential function.

One can also easily see that the π defined in (19c) is the ratio of the circumference of a circle to its diameter. For this purpose one extends the integral from $-\infty$ to ∞ and deforms the path of integration to a small circle about i; the integrand $dz/(z-i)(z+i)$ is easily seen to approach ds/r as the radius of the circle contracts.

NOTE ON A PAPER OF BAGCHI AND CHATTERJEE

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In a recent paper [1], Bagchi and Chatterjee showed that if $f(u)$ is analytic except for poles in the finite plane and satisfies

$$f(u+v)-f(u-v)=-\frac{f'(u)f'(v)}{(f(u)-f(v))^2},$$

then $f(u)=\wp(u)+C$, where $\wp(u)$ is the Weierstrass $\wp$-function. In the present note we obtain a somewhat similar result for the equation

$$f(u+v)-f(u-v)=\frac{2f'(u)f(v)}{1-k^2f^2(u)f^2(v)}, \tag{1}$$

which is satisfied by the Jacobi elliptic function $sn\ u$.

It is clear first by taking $u\to v$, that $f(0)$ is finite, and by taking $v\to 0$ it is seen that either $f(u)\equiv$constant or $f(0)=0$. Moreover if we divide both sides of (1) by $2v$ and let $v\to 0$ we infer $f'(0)=1$. Then for $u=0$, (1) becomes $f(v)-f(-v)=2f(v)$ so that $f(v)$ is an odd function. If we next interchange u and v, (1) becomes

$$f(u+v)+f(u-v)=\frac{2f(u)f'(v)}{1-k^2f^2(u)f^2(v)}. \tag{2}$$

Comparison of (1) and (2) yields the addition formula

$$f(u+v)=\frac{f(u)f'(v)+f'(u)f(v)}{1-k^2f^2(u)f^2(v)}.$$

Now following [1], put

$$f(u+v)+f(u-v)=2\sum_{r=0}^{\infty}\frac{v^{2r}}{(2r)!}f^{(2r)}(u);$$

and let

$$f(v) = v + \frac{1}{3!}av^3 + \cdots. \tag{3}$$

Substituting in (2) we get

$$f(u) + \tfrac{1}{2}v^2f''(u) + \cdots = \frac{f(u)(1 + \tfrac{1}{2}av^2 + \cdots)}{1 - k^2f^2(u)(v^2 + \cdots)}.$$

Comparing coefficients of v^2 in both members we get

$$\tfrac{1}{2}f''(u) = k^2f^3(u) + \tfrac{1}{2}af(u).$$

Multiplying by $f'(u)$ and integrating, we have

$$f'^2(u) = k^2f^4(u) + af^2(u) + C; \tag{4}$$

since $f'(0) = 1$, it follows that $C = 1$.

The coefficient a occurring in (3) is arbitrary. However by means of a linear transformation, (4) can be reduced to the familiar Legendre form (see, for example [2, Ch. 22]). Indeed if we put $g(x) = \lambda f(x/\lambda)$, $x = \lambda u$, then (4) becomes

$$g'^2(x) = \frac{k^2}{\lambda^4}g^4(x) + \frac{a}{\lambda}g^2(x) + 1. \tag{5}$$

Now if we choose λ so that

$$\frac{k^2}{\lambda^4} + \frac{a}{\lambda^2} + 1 = 0, \tag{6}$$

then (5) is in normal form and it is evident that

$$g(x) = sn(x, k^2/\lambda^4).$$

If we prefer we can think of λ as arbitrary and then a is determined by (6).

Thus we have proved the following

THEOREM. *If $f(u)$ is analytic except for poles in the finite plane and satisfies* (1), *then either $f(u) \equiv$ constant or*

$$f(u) = \frac{1}{\lambda}\, sn\,(\lambda u, k^2/\lambda^4),$$

where λ is arbitrary.

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A NOTE ON GAMBLING SYSTEMS AND BIRTH STATISTICS

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Let X_n denote the gain of a gambler at the nth of a series of games. We assume that the random variables X_n are independent and that

$$EX_n = 0, \qquad E|X_n| \leqq C < \infty \qquad (n = 1, 2, \cdots). \tag{1}$$

We also suppose that the gambler's opponent has unlimited capital and is willing to play indefinitely, and that the gambler has the option of ending the series of games at any time. He uses a "system" to decide when to do this, defined by a sequence of functions $f_n(x_1, \cdots, x_n)$, $(n=1, 2, \cdots)$, such that

$$f_n(x_1, \cdots, x_n) = 1 \text{ or } 0, \tag{2}$$

$$\text{if } f_n(x_1, \cdots, x_n) = 0 \text{ then for every } m > 0, \tag{3}$$

$$f_{n+m}(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+m}) = 0,$$

and plays or does not play the $(n+1)$th game according as $f_n(X_1, \cdots, X_n)=1$ or 0. By (3) the series of games ends with the first n such that $f_n(X_1, \cdots, X_n)=0$. We assume that the first game is played always, and set $f_0 \equiv 1$. The total gain of a gambler using this system is the random variable

$$W = \sum_{n=0}^{\infty} f_n(X_1, \cdots, X_n) \cdot X_{n+1}, \tag{4}$$

assuming that this series converges with probability 1, while the duration of play is the random variable

$$D = \sum_{n=0}^{\infty} f_n(X_1, \cdots, X_n). \tag{5}$$

THEOREM. *If* $ED < \infty$ *then* $EW = 0$.

Proof. If $ED < \infty$ then by (1),

$$\begin{aligned} \sum_{n=0}^{\infty} E\left|f_n(X_1, \cdots, X_n) \cdot X_{n+1}\right| &= \sum_{n=0}^{\infty} Ef_n(X_1, \cdots, X_n) \cdot E|X_{n+1}| \\ &\leqq C \cdot \sum_{n=0}^{\infty} Ef_n(X_1, \cdots, X_n) = C \cdot ED < \infty. \end{aligned} \tag{6}$$

It follows [1, page 34] that the series (4) converges with probability 1 and that

$$EW = \sum_{n=0}^{\infty} Ef_n(X_1, \cdots, X_n) \cdot EX_{n+1} = 0. \tag{7}$$

If the system is such that $ED = \infty$ then it may happen that $EW \neq 0$. For example, let $\Pr(X_n = -1) = \Pr(X_n = 1) = \frac{1}{2}$ for all n and let $f_n(x_1, \cdots, x_n) = 0$ if and only if for some integer p, $1 \leqq p \leqq n$, the relation $x_1 + \cdots + x_p = a$ holds,

where a is a fixed integer. The series of games stops as soon as the total gain of the gambler equals a. It is well known from the theory of random walk [2] that the probability is 1 that the equation $X_1 + \cdots + X_n = a$ will hold for some n. It follows that Pr $(W=a)=1$ and therefore $EW=a$. Of course, from this fact and the theorem just proved it follows that $ED=\infty$ if $a \neq 0$, which is also known. (In fact $ED=\infty$ even if $a=0$, and this follows from the fact that $ED=\infty$ for $a=\pm 1$.)

The foregoing considerations were suggested by an attempt [3] to explain an observed preponderance of male children in a certain group of families by the hypothesis that whereas the probability of a male birth may be approximately 1/2, male children are regarded as more desirable, so that parents are more likely to stop having children when a male child is born. Writing -1 for a female child and 1 for a male child, the excess W of male over female births for a given family will have the form (4), where again Pr $(X_n=-1)=\Pr(X_n=1)=\frac{1}{2}$. In this case, however, we might have $f_n(x_1, \cdots, x_n)=0$ if and only if at least one of the numbers $x_1, \cdots, x_n=1$, which would amount to stopping as soon as the first male child is born. For this particular system $ED=\sum_{n=1}^{\infty} n/2^n=2$ and hence, as for any system for which $ED<\infty$, $EW=0$. If the system described in the previous paragraph could be used for some $a \geqq 1$, then $EW=a>0$, but there are practical difficulties consequent on the fact that $ED=\infty$ for such a system.

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MATHEMATICAL NOTES

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DRESHER'S INEQUALITY

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Melvin Dresher has proved by an ingenious method, using moment-space theory,* the inequality which I shall call Dresher's inequality:

If $p \geqq 1 \geqq r \geqq 0$, $f(x) \geqq 0$, $g(x) \geqq 0$, *and* $\phi(x)$ *is a distribution function*§ *then*†

$$\left[\frac{\int [f(x)+g(x)]^p d\phi(x)}{\int [f(x)+g(x)]^r d\phi(x)}\right]^{1/(p-r)} \leqq \left[\frac{\int [f(x)]^p d\phi(x)}{\int [f(x)]^r d\phi(x)}\right]^{1/(p-r)} + \left[\frac{\int [g(x)]^p d\phi(x)}{\int [g(x)]^r d\phi(x)}\right]^{1/(p-r)}. \tag{1}$$

It is the purpose of this note to give an elementary proof of this inequality, based on the Minkowski inequality and an inequality due to Radon.‡ The Radon inequality is gotten easily by transforming the Hölder inequality;§§ thus Dresher's inequality is seen to be a mélange of the Hölder and Minkowski inequalities. It is appropriate to note here that Dresher's inequality is a generalization of an inequality due to Beckenbach.††

The Radon inequality is as follows:

If $\lambda > 0$, $\alpha_i \geqq 0$, $\beta_i > 0$, $i = 1, \cdots, n$, *then*

$$\frac{\left(\sum \alpha_i\right)^{\lambda+1}}{\left(\sum \beta_i\right)^{\lambda}} \leqq \sum \frac{\alpha_i^{\lambda+1}}{\beta_i^{\lambda}}. \tag{2}$$

In using this to prove Dresher's inequality, we shall need only two terms. The inequality is then stated as follows:

If $\lambda > 0$, $\alpha_1 \geqq 0$, $\alpha_2 \geqq 0$, $\beta_1 > 0$, *and* $\beta_2 > 0$, *then*

$$\frac{(\alpha_1+\alpha_2)^{\lambda+1}}{(\beta_1+\beta_2)^{\lambda}} \leqq \frac{\alpha_1^{\lambda+1}}{\beta_1^{\lambda}} + \frac{\alpha_2^{\lambda+1}}{\beta_2^{\lambda}}. \tag{3}$$

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* Forthcoming.

§ In particular, we may take $\phi(x) \equiv x$, and read each $d\phi(x)$ as dx.

† We assume that neither f nor g vanishes identically.

‡ J. Radon, Über die absolut additiven Mengenfunktionen, Wiener Sitzungsber. (II a), vol. 122, 1913, p. 1351. See also Hardy, Littlewood, and Polya, Inequalities, Cambridge, 1934, p. 61, problem 65, and C. B. Morrey, A class of representations of manifolds, Amer. J. Math. vol. 55, 1933, p. 692.

§§ Hardy, Littlewood, Polya, loc. cit.

†† E. F. Beckenbach, A class of mean value functions, this MONTHLY vol. 57, 1950, pp. 1–6.

We make in (3) the substitutions

$$\alpha_1 = \left[\int f^p d\phi\right]^{1/p} \qquad \beta_1 = \left[\int f^r d\phi\right]^{1/r}$$

$$\lambda = \frac{r}{p-r}.$$

$$\alpha_2 = \left[\int g^p d\phi\right]^{1/p} \qquad \beta_2 = \left[\int g^r d\phi\right]^{1/r}$$

Here, for the moment, we have assumed $p>r>0$.

Then the right side of (3) becomes the right side of (1), and the left side of (3) becomes

$$\frac{\left\{\left[\int f^p d\phi\right]^{1/p} + \left[\int g^p d\phi\right]^{1/p}\right\}^{p/(p-r)}}{\left\{\left[\int f^r d\phi\right]^{1/r} + \left[\int g^r d\phi\right]^{1/r}\right\}^{r/(p-r)}}. \tag{4}$$

But by Minkowski's inequality with $p \geqq 1$ and $0<r\leqq 1$ respectively, both

$$\left\{\left[\int f^p d\phi\right]^{1/p} + \left[\int g^p d\phi\right]^{1/p}\right\}^{p} \geqq \int (f+g)^p d\phi \tag{5}$$

and

$$\left\{\left[\int f^r d\phi\right]^{1/r} + \left[\int g^r d\phi\right]^{1/r}\right\}^{r} \leqq \int (f+g)^r d\phi, \tag{6}$$

so that the quantity (4) is not less than the expression on the left side of (1), and Dresher's inequality is proved for $p \geqq 1$ and $p>r>0$. Extension to the remaining cases is trivial, and thus the proof is complete.

THE COMPOSITION OF ROTATIONS IN EUCLIDEAN THREE-SPACE

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In the usual treatment by vectors of rotations about an axis in Euclidean space of three dimensions, it is proved that:

THEOREM 1. *If $\mathcal{R}_{\mathbf{u},\delta}$ represents the rotation about an axis through the origin parallel to the unit vector* $\mathbf{u}$, *through an angle* 2δ, *and* $\mathbf{x}$ *is the position vector of a general point, then*

$$\mathcal{R}_{\mathbf{u},\delta}\mathbf{x} = \mathbf{x} + 2[\mathbf{u} \times (\mathbf{u} \times \mathbf{x}) \sin\delta + \mathbf{u} \times \mathbf{x} \cos\delta] \sin\delta.$$

If $\delta \neq (2r+1)\pi/2$, the rotation is determined by the single vector $\boldsymbol{\lambda} = \mathbf{u} \tan\delta$, and, if we write $\mathcal{R}_{\boldsymbol{\lambda}} = \mathcal{R}_{\mathbf{u},\delta}$, Theorem 1 becomes:

THEOREM 2. $\mathcal{R}_{\boldsymbol{\lambda}}\mathbf{x} = \mathbf{x} + 2[\boldsymbol{\lambda} \times (\boldsymbol{\lambda} \times \mathbf{x}) + \boldsymbol{\lambda} \times \mathbf{x}]/(1+\boldsymbol{\lambda}^2)$.

The resultant of two rotations about concurrent axes is a rotation about an axis concurrent with them, so that we may write

$$\mathcal{R}\lambda_2\mathcal{R}\lambda_1 = \mathcal{R}\lambda,$$

and this note is concerned principally with the vector and matrix forms of the expression for λ in terms of λ_1 and λ_2.

For the vector form I prove:

THEOREM 3. *If* $\mathcal{R}\lambda_2\mathcal{R}\lambda_1=\mathcal{R}\lambda$, *then*

$$\lambda = (\lambda_1 + \lambda_2 - \lambda_1 \times \lambda_2)/(1 - \lambda_1 \cdot \lambda_2).$$

If the denominator vanishes, then the resultant rotation is through an angle π about an axis parallel to $\lambda_1+\lambda_2-\lambda_1\times\lambda_2$. The exceptional case of a rotation through π should be allowed for in all theorems which follow, but since it may always be dealt with as a limiting case, I do not make further reference to it.

Theorem 3 is a consequence of the theorems numbered 4 and 6 below. Theorem 4 might be called *the fundamental vector relations for a spherical triangle*, namely:

THEOREM 4. *ABC is a right-handed spherical triangle on a unit sphere center O. If the sensed segments OA, OB, OC represent the unit vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, *and* $\boldsymbol{\alpha}=\mathbf{u}\tan A$, $\boldsymbol{\beta}=\mathbf{v}\tan B$, $\boldsymbol{\gamma}=\mathbf{w}\tan C$, *then*

$$\begin{aligned}\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma} &= \boldsymbol{\beta}\cdot\boldsymbol{\gamma}\boldsymbol{\alpha} - \boldsymbol{\beta}\times\boldsymbol{\gamma}\\ &= \boldsymbol{\gamma}\cdot\boldsymbol{\alpha}\boldsymbol{\beta} - \boldsymbol{\gamma}\times\boldsymbol{\alpha}\\ &= \boldsymbol{\alpha}\cdot\boldsymbol{\beta}\boldsymbol{\gamma} - \boldsymbol{\alpha}\times\boldsymbol{\beta}.\end{aligned}$$

Let $\mathbf{u}'$, $\mathbf{v}'$, $\mathbf{w}'$ be the position vectors of the vertices of the polar triangle of ABC, so that we have the four sets, each of three relations:

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= \cos a,\\ \mathbf{v}\times\mathbf{w} &= \mathbf{u}'\sin a,\\ \mathbf{v}'\cdot\mathbf{w}' &= -\cos A,\\ \mathbf{v}'\times\mathbf{w}' &= \mathbf{u}\sin A.\end{aligned}$$

If we write $D_{\mathbf{x},\mathbf{y},\mathbf{z}}$ for $\mathbf{x}\cdot\mathbf{y}\times\mathbf{z}$, the identical relation connecting the four vectors $\mathbf{u}'$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is

$$D_{\mathbf{u},\mathbf{v},\mathbf{w}}\mathbf{u}' = D_{\mathbf{u}',\mathbf{v},\mathbf{w}}\mathbf{u} - D_{\mathbf{u}',\mathbf{u},\mathbf{w}}\mathbf{v} + D_{\mathbf{u}',\mathbf{u},\mathbf{v}}\mathbf{w},$$

where $D_{\mathbf{u},\mathbf{v},\mathbf{w}}=\sin A\sin b\sin c=\sin B\sin c\sin a=\sin C\sin a\sin b$, $D_{\mathbf{u}',\mathbf{v},\mathbf{w}}=\mathbf{u}'\cdot\mathbf{u}'\sin a=\sin a$, $D_{\mathbf{u}',\mathbf{u},\mathbf{w}}=-\mathbf{u}'\cdot\mathbf{v}'\sin b=\cos C\sin b$, and $D_{\mathbf{u}',\mathbf{u},\mathbf{v}}=\mathbf{u}'\cdot\mathbf{w}'\sin c=-\cos B\sin c$; so that

$$\mathbf{v}\times\mathbf{w}\sin B\sin c = \mathbf{u}\sin a - \mathbf{v}\cos C\sin b - \mathbf{w}\cos B\sin c,$$

i.e.,

$$\beta \times \gamma \cot B \cot C \sin B \sin c$$
$$= \alpha \cot A \sin a - \beta \cot B \cos C \sin b - \gamma \cot C \cos B \sin c,$$

whence the result follows, since

$$\frac{\cos A}{\cos B \cos C} = \cos a \tan B \tan C - 1 = \beta \cdot \gamma - 1.$$

A consequence of theorem 4 is

THEOREM 5. *If H is the orthocenter of the spherical triangle, then OH is parallel to $\alpha+\beta+\gamma$.*

For: $\alpha+\beta+\gamma$ is parallel to the plane containing α and $\beta\times\gamma$, that is, to that containing $\mathbf{u}$, and $\mathbf{u}'$, and this is the plane of the great circle through A and the pole of the great circle BC. Thus $\alpha+\beta+\gamma$ is parallel to the line of intersection of the three planes perpendicular to the side-arcs of the triangle and through the opposite vertices.

The other theorem required in the proof of theorem 3 is Donkin's theorem, namely

THEOREM 6. *ABC is a spherical triangle on a sphere center O. The resultant of a rotation through an angle $-2A$ about OA, followed by $-2B$ about OB, followed by $-2C$ about OC is the identity transformation.*

Denote by $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$ the reflection transformations in the three planes BOC, COA, AOB. Then the rotation through $-2A$ about OA is the resultant of $\mathfrak{S}_2$ followed by $\mathfrak{S}_3$, and similarly for the others, so that the resultant of the three rotations is

$$(\mathfrak{S}_3\mathfrak{S}_2)(\mathfrak{S}_1\mathfrak{S}_3)(\mathfrak{S}_3\mathfrak{S}_2) = \mathfrak{S}_2\mathfrak{S}_1^2\mathfrak{S}_3^2\mathfrak{S}_2 = \mathfrak{I}.$$

THEOREM 7. *If* $\mathbf{f}(\alpha, \beta) = (\alpha+\beta-\alpha\times\beta)/(1-\alpha\cdot\beta)$ *then* $\mathbf{f}[\mathbf{f}(\alpha, \beta), \gamma] = \mathbf{f}[\alpha, \mathbf{f}(\beta, \gamma)]$,

a result which may be readily verified by direct evaluation.

If an orthogonal 3-by-3 matrix represents a rotation (not through an odd multiple of π) about a line through the origin in Cartesian three-space, then it may be expressed in the standard Cayley form $(\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}$, where $\mathbf{A}$ is skew symmetric, say

$$\mathbf{A} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

It is to be proved that

THEOREM 8. *The rotation corresponding to* $\mathbf{A}$ *is about an axis parallel to the vector* (a_1, a_2, a_3) *through an angle* 2δ *given by* $\tan \delta = \sqrt{(a_1^2+a_2^2+a_3^2)}$.

In other words, (a_1, a_2, a_3) is precisely the vector $\mathfrak{a}$ used in Theorem 2. For convenience write $\mathbf{a}$ for $\mathfrak{a}$, and $t = \sqrt{(a_1^2+a_2^2+a_3^2)}$. In vector form, the transform of the point of position vector $\mathbf{x}$ by the rotation represented by $\mathbf{a}$ is

$$\bar{\mathbf{x}} = \mathbf{x} + 2[\mathbf{a} \times (\mathbf{a} \times \mathbf{x}) + (\mathbf{a} \times \mathbf{x})]/(1 + t^2).$$

Now $\mathbf{a} \times \mathbf{x} = \mathbf{A}\mathbf{x}$ (treating both as column-vectors) so that in matrix form this equation may be written

$$\bar{\mathbf{x}} = \left[\mathbf{I} + \frac{2}{1 + t^2}(\mathbf{A}^2 + \mathbf{A})\right]\mathbf{x};$$

also

$$\mathbf{A}^2 = \mathbf{a}\mathbf{a}^T - \mathbf{a}^T\mathbf{a}\mathbf{I},$$

where $\mathbf{a}\mathbf{a}^T$ is the dyadic product of the column vector $\mathbf{a}$ and its transpose, the row vector $\mathbf{a}^T$, *i.e.*, the 3-by-3 matrix with elements $a_i a_j$, and $\mathbf{a}^T\mathbf{a}$ is the scalar, or inner, product, *i.e.*, $\mathbf{a}^T\mathbf{a} = t^2$. To prove that this form is identical with the Cayley form, we have to prove that

$$\mathbf{I} + \mathbf{A} = \frac{1}{1 + t^2}[(1 + t^2)\mathbf{I} + 2(\mathbf{a}\mathbf{a}^T - t^2\mathbf{I} + \mathbf{A})](\mathbf{I} - \mathbf{A}).$$

The identity of the two expressions follows from the relations $\mathbf{a}^T\mathbf{A} = \mathbf{o}$, and $\mathbf{A}^2 = \mathbf{a}\mathbf{a}^T - t^2\mathbf{I}$.

Corresponding to Theorem 3 we have

THEOREM 9. *If* $\mathbf{L}$ *and* $\mathbf{M}$ *are two orthogonal 3-by-3 matrices, given by* $\mathbf{L} = (\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}$, $\mathbf{M} = (\mathbf{I}+\mathbf{B})(\mathbf{I}-\mathbf{B})^{-1}$, *and their product* $\mathbf{N} = \mathbf{M}\mathbf{L}$, *is given by* $\mathbf{N} = (\mathbf{I}+\mathbf{C})(\mathbf{I}-\mathbf{C})^{-1}$, *then* $\mathbf{C} = (\mathbf{A}+\mathbf{B}+\mathbf{A}\mathbf{B}-\mathbf{B}\mathbf{A})/[\det(\mathbf{I}+\mathbf{A}\mathbf{B})]^{1/2}$.

This theorem may be proved as a relation among skew-symmetric 3-by-3 matrices, but only by making use of their associated vectors, that is, not in such a way as would make possible any application of this result to matrices of other sizes. Posed in this way it constitutes quite a nice "Elementary Problem." More easily we may prove it directly from Theorem 3, by using the following relations:

(i) as a matrix, $-\mathbf{a} \times \mathbf{b}$ may be expressed as $\mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T$,

(ii) $\mathbf{A}\mathbf{B} = \mathbf{b}\mathbf{a}^T - \mathbf{a}^T\mathbf{b}\mathbf{I}$, so that

(iii) $\mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T = \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}$,

(iv) $\det(\mathbf{I}+\mathbf{A}\mathbf{B}) = \det[(1-\mathbf{a}^T\mathbf{b})\mathbf{I}+\mathbf{b}\mathbf{a}^T] = (1-\mathbf{a}^T\mathbf{b})^3+(1-\mathbf{a}^T\mathbf{b})^2\mathbf{a}^T\mathbf{b}$
$= (1-\mathbf{a}^T\mathbf{b})^2$,

(v) from Theorem 3, for the vectors associated with $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, we have

$$\mathbf{c} = (\mathbf{a} + \mathbf{b} - \mathbf{a} \times \mathbf{b})/(1 - \mathbf{a} \cdot \mathbf{b}).$$

At the referee's suggestion I add a note of how Theorems 3 and 8 appear at a slightly higher level of sophistication.

For Theorem 3 I use again $\boldsymbol{\lambda}$ for the vector representing a rotation, taking it now to have rectangular components λ, μ, ν. The rotation may also be represented by the quaternion $q=(1/\sqrt{1+l^2})\ (1+\lambda i+\mu j+\nu k)$. It is well-known that: *The resultant of the two rotations represented by the two quaternions q_1, q_2 is represented by the product of the quaternions*, namely, by

$$q_1q_2=\frac{1-\lambda_1\lambda_2-\mu_1\mu_2-\nu_1\nu_2}{\sqrt{\{(1+l_1^2)(1+l_2^2)\}}}\left\{1+i\,\frac{\lambda_1+\lambda_2+\mu_1\nu_2-\mu_2\nu_1}{1-\lambda_1\lambda_2-\mu_1\mu_2-\nu_1\nu_2}+j\cdots+k\cdots\right\}.$$

The vector corresponding to this quaternion is that given in Theorem 3.

Theorem 8 is another aspect of the well-known theorem (in the notation of Theorem 8): *The orthogonal matrix corresponding to the rotation represented by the vector* $\mathbf{a}$ *is* $\exp((2\delta/t)\mathbf{A})$. It is easily verified that

$$\exp\left(\frac{2\delta}{t}\mathbf{A}\right)=\mathbf{I}+\frac{1}{1+t^2}(\mathbf{A}+\mathbf{A}^2).$$

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

A CONNECTION BETWEEN TAYLOR'S THEOREM AND LINEAR DIFFERENTIAL EQUATIONS

D. C. LEWIS, Johns Hopkins University

The following amusing elementary derivation and generalization of Taylor's theorem (with the integral form for the remainder) was accidentally encountered while teaching a calculus course in which, for the convenience of the engineering students, the introduction of Taylor's theorem was postponed until after they had had some elementary theory of differential equations.

We consider the general homogeneous linear differential equation of the nth order with coefficients $a_1(x), \cdots, a_n(x)$ continuous on a certain interval I of the x-axis,

$$y^{(n)}+a_1(x)y^{(n-1)}+\cdots+a_{n-1}(x)y'+a_n(x)y=0, \tag{1}$$

and suppose we consider the system of independent solutions $y=y_k(x,\ t)$, $k=0, 1, 2, \cdots, n-1$, determined by the conditions

$$(d^k/dx^k)y_k(x,\ t)\,\big|_{x=t}=1, \qquad (d^h/dx^h)y_k(x,\ t)\,\big|_{x=t}=0, \qquad h\neq k. \tag{2}$$

Here the parameter t may take on any value in I. Then the solution of the corresponding non-homogeneous equation,

$$Y^{(n)} + a_1(x)Y^{(n-1)} + \cdots + a_n(x)Y = F(x) \tag{3}$$

(where $F(x)$ is continuous on I), satisfying the initial conditions

$$Y^{(k)}(c) = b_k, \quad k = 0, 1, 2, \cdots, n-1, c \in I, \tag{4}$$

can be written in the form

$$Y(x) = \sum_{k=0}^{n-1} b_k y_k(x, c) + \int_c^x F(t) y_{n-1}(x, t)dt. \tag{5}$$

This essentially well known result can be verified *a posteriori*, by differentiation of (5) and substitution in (3), using, among the other elements in the definition of the $y_k(x, t)$, the fact that $y_{n-1}(x, t)$ satisfies (1) for every value of t. It is also possible to deduce (5) as the end result of the Lagrange method of variation of parameters.

But *every* function $f(x)$ of class C^n on I can be regarded as being a solution of (3) no matter how the coefficients $a_1(x), \cdots, a_n(x)$ may have been chosen, so long as we take

$$F(x) = f^{(n)}(x) + \sum_{k=1}^{n} a_k(x) f^{(n-k)}(x). \tag{6}$$

Hence, with this definition of $F(x)$, we find that every function $f(x)$ of class C^n on I satisfies the identity,

$$f(x) = \sum_{k=0}^{n-1} f^{(k)}(c) y_k(x, c) + \int_c^x F(t) y_{n-1}(x, t)dt, \tag{7}$$

which we of course deduce from (5) by taking appropriate values for the b's.

Formula (7) may be regarded as a generalization of Taylor's theorem. In fact, if $a_1 \equiv a_2 \equiv \cdots \equiv a_n \equiv 0$, we must have, from (6), $F(x) = f^{(n)}(x)$, and, from (1) and (2), we have

$$y_k(x, t) = (x - t)^k/k! \qquad k = 0, 1, \cdots, n-1.$$

Hence (7) becomes in this special case

$$f(x) = \sum_{k=0}^{n-1} f^{(k)}(c) \frac{(x-c)^k}{k!} + \frac{1}{(n-1)!} \int_c^x f^{(n)}(t)(x-t)^{n-1}dt,$$

which is Taylor's theorem with integral form of the remainder, usually proved directly by an inductive process employing integration by parts.

Obviously there are many other special cases of (7) which can be obtained by special choices of the a's.

INVERSE SQUARE ORBITS. A SIMPLE TREATMENT

D. E. RICHMOND, Williams College

The determination of the orbit of a body attracted to a center by an inverse square force is of great historical importance. Its rôle in astronomy and atomic physics makes the derivation of continuing interest. However, it is usually accessible to the student only after a fairly extensive knowledge of differential equations. There are available a number of vector treatments but they too require a considerable degree of mathematical maturity for their understanding.

This paper presents an extremely simple vector derivation of the equation of the orbit and of the relation between velocity and radial distance. It also includes a simple construction for the velocity. This treatment presupposes a knowledge of vector differentiation but uses neither scalar nor vector products. It is believed that the approach is well within the understanding of a college sophomore.

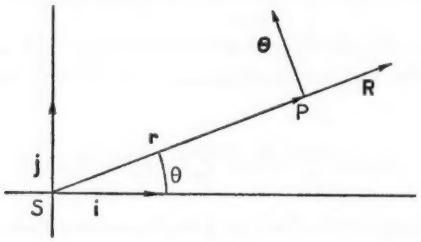

FIG. 1

Let a planet P be located at any time t by the vector $\mathbf{r}(t)$ from the sun S to P. Let $r(t)$ be the length of $\mathbf{r}(t)$ and let $\theta(t)$ specify its direction. Hereafter, the argument t will be omitted. In terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$,

$$\mathbf{r} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = r\mathbf{R},$$

where $\mathbf{R}$ is a unit vector in the direction of $\mathbf{r}$.

The velocity $\mathbf{v} = d\mathbf{r}/dt$, hence

$$\mathbf{v} = r'(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + r(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\theta' = r'\mathbf{R} + r\theta'\mathbf{\Theta} \tag{1}$$

where $\mathbf{\Theta}$ is a unit vector perpendicular to $\mathbf{R}$ in the direction of increasing θ, and where primes denote differentiation with respect to t.

Implicit in (1) is the relation $d\mathbf{R}/dt = \theta'\mathbf{\Theta}$. A companion relation is $d\mathbf{\Theta}/dt = -\theta'\mathbf{R}$, which is immediately verified by differentiating $\mathbf{\Theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$.

One other preliminary is necessary. For central forces, $r^2\theta' = h$, a constant. The derivation is familiar and will be be omitted here.

The inverse square law is expressed by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{-\mu}{r^2}\mathbf{R} \tag{2}$$

where μ is a constant. To integrate (2), it is desirable to express the right member as a time derivative. This suggests substituting $\mathbf{R} = -(1/\theta')d\mathbf{\Theta}/dt$. Using $r^2\theta' = h$,

$$\frac{d\mathbf{v}}{dt} = \frac{-\mu}{r^2}\left(-\frac{1}{\theta'}\frac{d\mathbf{\Theta}}{dt}\right) = \frac{\mu}{r^2\theta'}\frac{d\mathbf{\Theta}}{dt} = \frac{\mu}{h}\frac{d\mathbf{\Theta}}{dt}. \tag{3}$$

Then

$$\frac{h}{\mu}\mathbf{v} = \mathbf{\Theta} + \mathbf{C} \tag{4}$$

where $\mathbf{C}$ is a constant vector.

The value of $\mathbf{C}$ depends upon the initial conditions. Let us assume that at $t=0$, P is closest to S, that is, at perihelion, and that $\mathbf{i}$ is drawn in this direction. Since $r'=0$ at $t=0$, $\mathbf{v}_0$ is in the direction of $\mathbf{\Theta}_0=\mathbf{j}$. Hence $\mathbf{C}=\epsilon\mathbf{j}$ where ϵ is constant. Thus

$$\frac{h}{\mu}\mathbf{v} = \mathbf{\Theta} + \epsilon\mathbf{j}. \tag{5}$$

Figure 2 gives the corresponding geometrical construction.

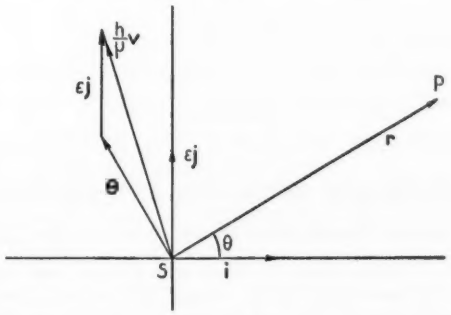

FIG. 2

Let us now take components of the vector $(h/\mu)\mathbf{v}$ in the direction of $\mathbf{\Theta}$. The result is $(h/\mu)r\theta'$ or, since $r^2\theta' = h$, $h^2/(\mu r)$. By (5), this must equal the projection of $\mathbf{\Theta}+\epsilon\mathbf{j}$ on $\mathbf{\Theta}$, namely, $1+\epsilon\cos\theta$. Hence

$$\frac{h^2}{\mu r} = 1 + \epsilon\cos\theta \tag{6}$$

or

$$r = \frac{p}{1 + \epsilon \cos \theta}, \qquad p = \frac{h^2}{\mu}, \tag{7}$$

which is the polar equation of a conic. We have assumed that r is a minimum at $\theta = 0$. This implies that $\epsilon > 0$, excluding the trivial case of the circular orbit ($\epsilon = 0$).

From the law of cosines applied to the triangle of sides Θ, $\epsilon\mathbf{j}$, and $(h/\mu)\mathbf{v}$ (figure 2),

$$\frac{h^2 v^2}{\mu^2} = 1 + \epsilon^2 + 2\epsilon \cos \theta.$$

Substituting for $\epsilon \cos \theta$ from (6)

$$v^2 = \frac{2\mu}{r} - \frac{\mu^2(1 - \epsilon^2)}{h^2}.$$

In the important case of an elliptical orbit ($\epsilon < 1$), the semi-major axis

$$a = \frac{p}{1 - \epsilon^2} = \frac{h^2}{\mu(1 - \epsilon^2)}.$$

Then

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right], \tag{8}$$

a widely used relation.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separated, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1041. *Proposed by SHE SLID and O PARISH, Massachusetts Institute of Technology*

Surely the days of anagrams are not dead. Here are four well known mathe-

maticians: (1) A DONUT SHIP, (2) SHE IS A NUT, (3) SEWER STAIRS, (4) FIRE ON SUB.

E 1042. *Proposed by P. L. Chessin, Cooper Union*

A sequence $\{a_n\}$, where a_0 and a_1 are given numbers, is defined recursively by $a_{n+2}=(a_{n+1}+a_n)/2$. Find $\lim_{n\to\infty} a_n$.

E 1043. *Proposed by O. J. Ramler, Catholic University, Washington, D.C.*

Prove that the sum of the ratios in which a point within a triangle divides the Cevians of this point is never less than 6 and that the product of the ratios is never less than 8.

E 1044. *Proposed by J. E. Wilkins, Jr., Nuclear Development Associates, Inc.*

Find

$$\prod_{r=1}^{m-1}\left(2\sin\frac{\pi r}{m}\right)^r.$$

E 1045. *Proposed by S. W. Golomb, Harvard University*

Let $n\phi$ (pronounced "n-phi-torial") denote the product of all integers up to n which are prime to n. Then $n\phi \equiv \pm 1 \pmod{n}$. Prove this, and determine when the value is $+1$ and when it is -1.

SOLUTIONS

Rectification of the Loop of the Folium of Descartes

E 1009 [1952, 181]. *Proposed by Arthur Danzl, St. John's University*

Rectify the loop of the folium of Descartes by means of elementary functions either finitely, if possible, or approximately.

I. *Solution by Julian Braun, Washington, D.C.* Parametric equations of the folium of Descartes are

$$x = 3at/(1+t^3), \qquad y = 3at^2/(1+t^3).$$

Then

$$s = \int_A^B (dx^2+dy^2)^{1/2} = 3a\int_0^\infty \frac{[(1-2t^3)^2+(2t-t^4)^2]^{1/2}}{(1+t^3)^2}\,dt$$

$$= 3a\int_0^\infty f(t)dt = 6a\int_0^1 f(t)dt.$$

We now construct the following table:

t	$f(t)$	t	$f(t)$	t	$f(t)$
0	1.00000000	9/24	1.04194530	17/24	0.65336168
1/24	1.00317665	10/24	1.02036581	18/24	0.59051585
2/24	1.01147315	11/24	0.98916542	19/24	0.53185302
3/24	1.02292811	12/24	0.94861100	20/24	0.47940652
4/24	1.03545699	13/24	0.89957859	21/24	0.43477191
5/24	1.04690976	14/24	0.84349983	22/24	0.39892511
6/24	1.05515547	15/24	0.78226362	23/24	0.37206317
7/24	1.05818861	16/24	0.71808575	24/24	0.35355339
8/24	1.05424911				

Applying Simpson's rule, Weddle's rule, and Newton's sixth degree integration formula to the data in the above table we obtain

$$s \doteq 4.9174888a, \quad 4.9174889a, \quad 4.9174887a$$

respectively. Hence we conclude that, to six decimal places,

$$s = 4.917489^{-}a.$$

II. *Remarks by O. E. Stanaitis, St. Olaf College.* We observe that

$$s = 6a\int_0^1 \frac{\sqrt{1+tf(t)}}{(1+t^3)^2}\,dt,$$

where $f(t)=4t-4t^2-4t^4+4t^5+t^7$. It is easily proved that $f(t)<1$ for $0\leqq t<1$ and $f(1)=1$. Expansion by the binomial formula for $0\leqq t<1$ gives

$$\sqrt{1+tf(t)} = \sum_{n=0}^{\infty}\binom{1/2}{n}t^n[f(t)]^n.$$

Since $\binom{1/2}{n}=O(n^{-3/2})$ it follows that the series is absolutely convergent in the closed interval $0\leqq t\leqq 1$, and therefore

$$s = 6a\sum_{n=0}^{\infty} g_n, \qquad g_n = \binom{1/2}{n}\int_0^1 \frac{t^n[f(t)]^n}{(1+t^3)^2}\,dt.$$

The integrals are of a well known type (*cf.*, Rinehart Math. Tables, Nos. 235, 247, 248) and may be easily calculated. If we notice that the expression for s is an alternating series and therefore the absolute value of the remainder r_n is less than the absolute value of the first neglected term, we have

$$|r_n| < |g_{n+1}| < \left|\binom{1/2}{n+1}\right|\int_0^1 t^{n+1}dt = \left|\binom{1/2}{n+1}\right|\frac{1}{n+2}.$$

Thus if 3 terms of the series are taken $|r_3|<2^{-7}$.

Also solved by the proposer.

Covering by Dissected Squares

E 1010 [1952, 181]. *Proposed by Michael Goldberg, Washington, D. C.*

Given n equal unit squares, dissect each of these in exactly the same way with straight cuts into $p(n)$ parts such that the n $p(n)$ pieces may be assembled to form a square of edge $\sqrt{n}$. Investigate $k(n)$, the minimal values of $p(n)$, and show that $k(n) \leqq 5$ for all n.

Solution by the Proposer. Let $[\sqrt{n}]$ be the integral part of $\sqrt{n}$. Then if $\sqrt{n}-[\sqrt{n}]>1/2$, take $[\sqrt{n}]$ as the edge of a basic square. If $\sqrt{n}-[\sqrt{n}]<1/2$, take $[\sqrt{n}]-1$ as the edge of a basic square. Then if the basic square is placed in the corner of a square of edge $\sqrt{n}$, as shown in Fig. 1, it will be bordered by an L-shaped region of width w. This width is restricted by the relation $3/2>w>1/2$. By a single cut, this L-shaped region can be changed into a rectangle of width w.

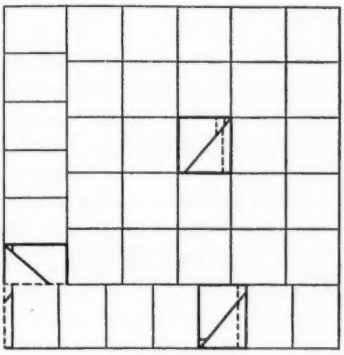

FIG. 1

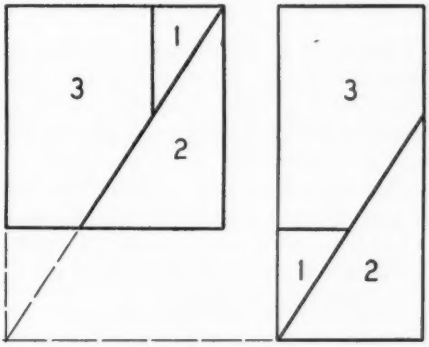

FIG. 2

Fill the basic square with the required number of given unit squares. The remainder of the unit squares must now be transformed by dissection into rectangles of width w. Since $3/2>w>1/2$, this transformation never requires more than three pieces, as shown by the solid lines. (See separate demonstration of this construction.) The dotted lines forming the required cut in the rectangle of width w to transform it back into the L-shaped region will, with proper orientation of the division of a fundamental rectangle, divide one of the fundamental rectangles into, at most, five pieces. All other fundamental rectangles and squares should then be cut into the same five pieces.

Dissection of the unit square: A unit square can be cut into three pieces to form a rectangle of width w, where $3/2>w\geqq 1$, as shown in Fig. 2. The top edge of piece 1 is never more than half the horizontal dimension of the rectangle. Therefore, a vertical cut dividing this rectangle into any desired ratio can always be selected so that it cuts pieces 2 and 3 without cutting piece 1. Therefore, only five pieces will be formed.

If $1 \geqq w > 1/2$, the same dissection is used, except that w is now the short edge of the rectangle. A horizontal cut dividing the rectangle into any desired ratio can always be selected so that piece 1 is not cut. Therefore, at most, only five pieces will be formed. If only piece 3 is cut, then only four pieces will be formed.

Note: In a paper by the proposer and Dr. B. M. Stewart, now awaiting publication, the bound of this problem has been improved by showing that $k(n) \leqq 4$ for all n. In particular, $k(ab) \leqq 3$, where a and b are integers satisfying $b < a < 4b$; $k(4a) \leqq 3$; $k(4a+1) \leqq 3$; $k(a^2+b^2) \leqq 3$; $k[a^2+(a-b)^2] \leqq 2$ for $a \geqq 4b-1$; $k(2) = k(5) = k(10) = 2$.

Traffic Signals

E 1011 [1952, 249]. *Proposed by D. D. Wall, Los Angeles, California*

If traffic signals are "set for 30," at what other constant speeds may one travel? Assume signals evenly spaced, all change at the same time, and that they alternate in color spacewise as well as timewise.

Solution by C. A. Swanson, University of British Columbia. Let the distance between successive lights be d, so that the time between two changes of a signal is $T = d/30$. Suppose a car passes the first light at time $t = 0$; it must then pass the second light between times $(2k-1)T$ and $2kT$, $\cdots$, the nth light between times $(n-1)(2k-1)T$ and $[(n-1)(2k-1)+1]T$, where $k = 1, 2, \cdots$. Let the car pass the second light at time $(2k-1)T+\epsilon$, so that the constant velocity of the car is $v = d/[(2k-1)T+\epsilon]$. Since v is constant, the car passes the third light at time $2d/v = 2[(2k-1)T+\epsilon]$, $\cdots$, the nth light at time $(n-1)[(2k-1)T+\epsilon]$. In order that the car not stop at the nth light, or anywhere before the nth light,

$$(n-1)[(2k-1)T+\epsilon] < [(n-1)(2k-1)+1]T,$$

or

$$\epsilon < T/(n-1).$$

Thus

$$v = d/[(2k-1)T+\epsilon] \geqq 30/[(2k-1)+1/(n-1)],$$

equality being attained in the limit as $n \to \infty$. We therefore have

$$30/[(2k-1)+1/(n-1)] \leqq v \leqq 30/(2k-1). \tag{1}$$

Now suppose the car passes the first light at time T, the second light at time $2kT-\epsilon$, $\cdots$, the nth light at time $nkT-(n-1)\epsilon$, so that we must have

$$nkT-(n-1)\epsilon > (n-1)T,$$

or

$$\epsilon < T/(n-1).$$

Then

$$30/(2k-1) \leqq v \leqq 30/[(2k-1) - 1/(n-1)]. \tag{2}$$

Combining (1) and (2) we have, for the car's possible constant velocities,

$$30/[(2k-1) + 1/(n-1)] \leqq v \leqq 30/[(2k-1) - 1/(n-1)]. \tag{3}$$

If n is taken as infinite then result (3) reduces to

$$v = 30/(2k-1) = 30, 10, 6, \cdots .$$

Also solved by Julian Braun, J. L. Hildebrand, Vern Hoggatt, A. R. Hyde, David Mandelbaum, W. H. Mead, Jr., D. V. Newton, C. S. Ogilvy, R. Z. Vause, Jr., and the proposer. Most of these solutions assumed an infinite number of signals.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey, All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4513. *Proposed by Leonard Carlitz, Duke University*

Put $\{m\}! = (x^m - 1)(x^{m-1} - 1) \cdots (x - 1)$, $\{0\}! = 1$.

A. Show that

$$\frac{\{2m\}!\{2n\}!}{\{m+n\}!\{m\}!\{n\}!} \qquad (m, n \geqq 0) \tag{1}$$

is a polynomial with rational coefficients.

B. The quotient

$$\frac{\{mn\}!}{\{m\}!(\{n\}!)^m}(x-1)^m \tag{2}$$

is not integral for all $m, n \geqq 0$. How can it be modified to be integral and at the same time reduce to $(mn)!/m!(n!)^m$ for $x = 1$?

4514. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

$\psi(t)=$constant and $\psi(t)=|t|$ are solutions of the integral equation

$$2\psi(t)=\int_{-1}^{1}\psi(2tx)dx,$$

as is easily verified. Are there any other solutions?

4515. *Proposed by Karl Goldberg, Brooklyn, N.Y.*

Prove

$$\sum x \equiv 2^{-4}+(2^{p-1}-1)/p \pmod{p},$$

where p is a prime and the summation extends over all integers $x<p$ which are greater than $(p-1)/2$ and whose inverses mod p are also greater than $(p-1)/2$.

4516. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$ let L be the second Lemoine point (*i.e.* the point whose distances from the planes of the faces are proportional to the circumradii of these faces) and let L', L'', L''' be the harmonic conjugates of L with respect to the points where lines through L cut the edges BC and DA, CA and DB, AB and DC. Show (1) the tetrahedron $LL'L''L'''$ is self-conjugate with respect to the sphere $ABCD$; (2) the polar planes of L, L', L'', L''' with respect to the sphere $ABCD$ coincide with the polar planes of these points with respect to the tetrahedron $ABCD$.

4517. *Proposed by George Lorentz, University of Toronto*

A sequence a_n, $n=0, 1, \cdots$, is called logarithmically concave if

$$a_n^2-a_{n-1}a_{n+1}\geqq 0, \qquad n=1, 2, \cdots.$$

Prove that if the sequences a_n, b_n are positive and logarithmically concave, then their convolution $c_n=a_0b_n+a_1b_{n-1}+\cdots+a_nb_0$ has the same properties.

SOLUTIONS

Integral Reducing to an Elementary Function

4450 [1951, 495]. *Proposed by R. M. Redheffer, University of California, Los Angeles*

Let $g(x)$ be a polynomial of degree $\leqq 2n+1$. Show that a necessary and sufficient condition that the integral

$$\int e^{x^2}g(x)dx$$

is an elementary function is the vanishing of the determinant

$$\begin{vmatrix} g(0) & 1 & 0 & 0 \cdots & 0 & 0 \\ g^{(2)}(0) & 4 & 1 & 0 \cdots & 0 & 0 \\ g^{(4)}(0) & 0 & 8 & 1 \cdots & 0 & 0 \\ g^{(6)}(0) & 0 & 0 & 12 \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g^{(2n-2)}(0) & 0 & 0 & 0 \cdots & 4n-4 & 1 \\ g^{(2n)}(0) & 0 & 0 & 0 \cdots & 0 & 4n \end{vmatrix}.$$

Solution by P. Somanatham, M. R. College, Vizianagram, South India. Let

$$g(x) = \sum_{i=0}^{2n+1} A_i x^i.$$

It is obvious that

$$\int xe^{x^2}dx, \quad \int x^3e^{x^2}dx, \cdots, \quad \int x^{2r+1}e^{x^2}dx$$

are elementary functions. So it suffices to prove that the necessary and sufficient condition that the integral

$$\int (A_0 + A_2x^2 + \cdots + A_{2n}x^{2n})e^{x^2}dx \tag{1}$$

may be an elementary function is the vanishing of the given determinant. Now it is well known that $\int e^{x^2}dx$ is not elementary. Further

$$\int A_2x^2e^{x^2}dx = A_2(x/2)e^{x^2} - \int \tfrac{1}{2}A_2e^{x^2}dx,$$

$$\int A_4x^4e^{x^2}dx = A_4(x^3/2)e^{x^2} - \int (3/2)A_4x^2e^{x^2}dx$$

$$= A_4(x^3/2)e^{x^2} - 3A_4(x/4)e^{x^2} + \int (3/4)A_4e^{x^2}dx;$$

and so on. Hence the necessary and sufficient condition that (1) may be an elementary function is that

$$\left(A_0 - \frac{A_2}{2} + \frac{3A_4}{4} - \frac{3\cdot 5A_6}{8} + \cdots\right)e^{x^2}$$

should vanish, i.e., the expression in the brackets should be zero. Since $g(0) = A_0$, $g^{(2)}(0) = 2!\,A_2$, $g^{(4)}(0) = 4!\,A_4$, $\cdots$, the necessary and sufficient condition is the

vanishing of

$$g(0) - \frac{g^{(2)}(0)}{2\cdot 2!} + \frac{3g^{(4)}(0)}{4\cdot 4!} - \frac{3\cdot 5g^{(6)}(0)}{8\cdot 6!} + \cdots$$

or (upon multiplying by $4\cdot 8\cdot 12\cdots 4n$) the vanishing of

$$4\cdot 8\cdot 12\cdots 4n\cdot g(0) - 8\cdot 12\cdots 4n\ g^{(2)}(0) + 12\cdot 16\cdots 4n\cdot g^{(4)}(0) - \cdots$$

which is exactly the expanded form of the given determinant.

Also solved by A. S. Hendler, M. S. Klamkin, M. S. Robertson, and the Proposer.

Modulus of an Analytic Function

4451 [1951, 496]. *Proposed by Paul Erdös, University College, London, England*

Let

$$f(z) = \prod_{j=1}^{n} (z - z_j), \qquad |z_j| = 1.$$

Prove that one can connect the origin with a point of the unit circle, so that everywhere on this path, except at the origin, $|f(z)| < 1$.

Solution by Paul Cohen, Brooklyn College, New York. We assume merely that $f(z)$ is non-constant and analytic for $|z| \leqq 1$, that $|f(0)| = 1$ and that $f(z) \neq 0$ for $|z| < 1$.

Let $r < 1$. $1/f(z)$ is regular in $|z| \leqq r$ and attains its maximum modulus for $|z| \leqq r$ at some point ζ, $|\zeta| = r$. Thus $|f(\zeta)| < 1$. Let $z = re^{i\theta}$. By the continuity of $|f(z)|$ it follows that for each $r < 1$ there is an interval of θ, $\theta_1(r) < \theta < \theta_2(r)$ throughout which $|f(z)| < 1$.

Now the maximum modulus theorem applied to $f(z)$ in $|z| \leqq r$ assures that $|f(z)| > 1$ for some point on $|z| = r$. Hence we deduce the existence of θ_1 and θ_2 such that

$$|f(re^{i\theta_1})| = |f(re^{i\theta_2})| = 1.$$

The curves defined by $z = re^{i\theta_1}$ and $z = re^{i\theta_2}$, as r varies, extend all the way from the origin to the unit circle, and they do not meet. Otherwise they would bound a region having no zeros and on whose boundary $|f(z)| = 1$, but this is seen to be impossible by the maximum modulus theorem applied to $1/f(z)$. It is now evident that the required path can be traced from origin to unit circle with $\theta(r)$ always between $\theta_1(r)$ and $\theta_2(r)$.

Comments by the Proposer. Similarly one can connect the origin with a point of the unit circle by a path on which everywhere, except at the origin, $|f(z)| > 1$. To the conjecture that there exists a radius on which $|f(z)| < 1$, Charles Loewner has found a counter-example. The counter-example is not simple and it is not

given explicitly. The interesting problem remains: find an elementary counter-example.

The question then rises: Does there exist a constant c such that one can connect the origin with the unit circle by a path of length $<c$ so that on this path $|f(z)|<1$ and c is independent of $f(z)$?

A Number Theory Identity

4452. [1951, 496]. *Proposed by Leo Moser, University of Alberta, Canada*

Prove that

$$\sum_{r=1}^{n} \mu(r)[n/r]\{3-2\omega(r)\}^2 = 9 + 8\{\Pi_1^*(n) + \Pi_2^*(n)\},$$

where $\mu(r)$ is the Möbius function, $[n/r]$ denotes the largest integer not exceeding n/r. $\omega(r)$ is the number of distinct prime divisors of r (i.e., $\omega(1)=0$, $\omega(p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k})=k$), and $\Pi_k^*(n)$ is the number of integers t, not exceeding n, for which $\omega(t)=k$.

Solution by E. M. Wright, University of Aberdeen, Scotland. Let

$$m = p_1^{a_1}\cdots p_k^{a_k} \qquad (p_1 < p_2 < \cdots < p_k)$$

so that $\omega(m)=k$, and write $m'=p_1\cdots p_k$. Now $\mu(r)=0$ or $=(-1)^{\omega(r)}$ according as r has, or has not, a squared factor. Again, the number of divisors r of m' for which $\omega(r)=j$ is $\binom{k}{j}$. Hence

$$\sum_{r|m} \mu(r)x^{\omega(r)} = \sum_{r|m'} \mu(r)x^{\omega(r)} = \sum_{j=0}^{k} (-1)^k \binom{k}{j} x^j = (1-x)^k.$$

Differentiating s times with respect to x and putting $x=1$, we have

$$\sum_{r|m} \mu(r)T_s(r) = \begin{cases} (-1)^k k! & (s=k), \\ 0 & (s\neq k), \end{cases}$$

where

$$T_s(r) = \omega(r)\{\omega(r)-1\}\cdots\{\omega(r)-s+1\}.$$

Hence

$$(-1)^k k!\Pi_k^*(n) = \sum_{m=1}^{n}\sum_{r|m} \mu(r)T_k(r) = \sum_{r=1}^{n} \mu(r)[n/r]T_k(r),$$

where $\Pi_0^*(n)=1$. The required result follows at once, since

$$9T_0 - 8T_1 + 4T_2 = (3-2\omega)^2.$$

Also solved by Paul Cohen, Emil Grosswald, H. T. LaBorde and the Proposer.

A Summation Suggested by Certain Results of Ramanujan

4453 [1951, 569]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$\frac{1^5}{e^{\pi}+1}+\frac{3^5}{e^{3\pi}+1}+\frac{5^5}{e^{5\pi}+1}+\cdots=\frac{31}{504}.$$

Solution by Leonard Carlitz, Duke University. This solution was suggested by the Proposer's paper, *Three summations due to Ramanujan*, Quarterly Journal of Mathematics, Oxford, (2), I(1950), pp. 238–240. We have the known formula

$$\sum_{m=0}^{\infty} e^{-(2m+1)\pi x}=\frac{1}{e^{\pi x}-e^{-\pi x}}=\frac{1}{2\pi x}+\frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{x+ni}+\frac{1}{x-ni}\right\}.$$

Differentiating five times we get

$$\pi^5\sum_{m=0}^{\infty}(2m+1)^5e^{-(2m+1)\pi x}=\frac{1}{2\pi}\frac{5!}{x^6}+\frac{5!}{2\pi}\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{(x+ni)^6}+\frac{1}{(x-ni)^6}\right\}.$$

Next multiply both members by $(-1)^{x-1}$ and sum for x integral from $x=1$ to $x=\infty$; thus

$$\pi^5\sum_{m=0}^{\infty}\frac{(2m+1)^5}{e^{(2m+1)\pi}+1}=\frac{5!}{2\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^6}-\frac{5!}{2\pi}\sum_{m,n=1}^{\infty}(-1)^{m+n}\left\{\frac{1}{(m+ni)^6}+\frac{1}{(m-ni)^6}\right\}.$$

Note that interchange of m and n in the double sum changes the sign of each sum; hence the double sum vanishes. Finally we use the familiar formula

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^{2k}}=(-1)^{k-1}\frac{2^{2k-1}-1}{(2k)!}B_{2k}\pi^{2k},$$

where $B_2=1/6$, $B_4=1/30$, $B_6=1/42$, $\cdots$ are the Bernoulli numbers. Hence

$$\sum_{m=0}^{\infty}\frac{(2m+1)^5}{e^{(2m+1)\pi}+1}=\frac{5!}{2\pi^6}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^6}=\frac{5!}{2\pi^6}\frac{31}{6!}B_6\pi^6=\frac{31}{504}.$$

The same method, using $4k+1$ differentiations, yields

$$\sum_{m=0}^{\infty}\frac{(2m+1)^{4k+1}}{e^{(2m+1)\pi}+1}=\frac{2^{4k+1}-1}{8k+4}B_{4k+2}.$$

Also solved by T. M. Apostol, Joshua Barlaz, L. Carlitz (a second solution), J. S. Frame, Stanley Katz and A. M. Peiser, O. E. Stanaitis, Ernest Trost, R. E. Wild, and the Proposer.

Functions such that the Derivative of the Quotient is the Quotient of the Derivatives

4455 [1951, 569]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

If

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)}{g'(x)},$$

solve for $f(x)$ in terms of $g(x)$, and for $g(x)$ in terms of $f(x)$. (This is an extension of one of the problems proposed for the 1951 William Lowell Putnam Prize Competition.)

Solution by E. J. Scott, University of Illinois. Using the abbreviations f, g, f', g', for $f(x)$, $g(x)$, *etc.*, we have

$$(gf' - fg')/g^2 = f'/g', \tag{1}$$

from which

$$f'/f - g'/g = [1 - g/g']^{-1}.$$

Integrating, we have

$$\ln f/g = \int [1 - g/g']^{-1}dx,$$

whence

$$f = g \exp\left\{\int [1 - g/g']^{-1}dx\right\}.$$

Similarly, from (1) we obtain

$$(g'/g)^2 - (f'/f)(g'/g) + f'/f = 0,$$

whence

$$g'/g = \tfrac{1}{2}[f'/f \pm \sqrt{(f'/f)^2 - 4(f'/f)}].$$

Integrating and solving for g, we get

$$g = \sqrt{f} \exp\left[\pm \frac{1}{2}\int \sqrt{(f'/f)^2 - 4(f'/f)}\,dx\right].$$

Also solved by Emil Grosswald, Hyman Kaufman, R. A. Miller, George Millman, C. S. Ogilvy, Azriel Rosenfeld, O. E. Stanaitis, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Elementary Analysis. By K. O. May. New York, John Wiley and Sons, Inc. 1952. xviii+635 pages (including tables). $5.00.

This book is a unified treatment of algebra, trigonometry, analytic geometry and some of the concepts of calculus. It is by no means just another book in this field but is a thoroughly wrought piece of text-book writing. At least one year of high school algebra and plane geometry are supposed. The book is adaptable to students wishing a thorough grounding for calculus and beyond and to students wishing a terminal course. This last class of students can get some of the important ideas of mathematics which have given the subject its use and beauty. The author gives the teacher useful suggestions in making up a course.

The first three chapters are called: (1) Introduction, (2) Logic, (3) Numbers and Elementary Operations. They present topics such as the basic concept of a set and the construction of the number system from the positive integers. With a class of interested and well prepared students discussion of this material would be excellent and very profitable. If student preparation warrants, this material could be used otherwise or omitted and a course begun with chapter four. A good small bibliography of further readings for interested students is included.

The more traditional part of the book discusses the linear, quadratic, and power functions as well as the exponential, logarithmic, and circular functions. There are also chapters on analytic geometry of two and three dimensions, polynomials, the conics, complex numbers and algebraic functions with graphs of irrational functions. Parametric equations and polar coordinates are handled to give the student the necessary ideas for later courses. Tables occur at the end of the book.

It is clear that the author has spent both time and care in getting together a great many problems chosen to gain manipulative practice and illustrate the text material. The book is a vast store house of problems well chosen from the uses of mathematics in the physical sciences, economics and business. A teacher using the book should be on the lookout for the many exercises extending the theory. Included among such are important loci on polar coordinates and problems on various properties of the conics. The problems are graded in difficulty and more difficult problems are specially indicated. The reviewer would like to see more answers to problems given.

The author's style is lively and clear and his exposition excellent. The set-up of the book and the typography are very good. The inclusion in problem 14 on page 185 of information not needed to solve the problem may have been intentional.

Each college and university offering a unified Freshman course has its own particular problems. The reviewer believes that careful consideration of various courses designed from this well-written book may help to find a solution of such problems and to provide a first course in college mathematics which is beneficial to students and enjoyable to teachers.

W. N. HUFF
University of Oklahoma

Teaching Mathematics in the Secondary School by L. B. Kinney and C. R. Purdy. New York, Rinehart and Company, Inc. 1952. xviii+381 pages. $5.00.

This book is designed to furnish the teacher of mathematics information about what mathematics should be taught in the secondary school and how it should be taught. The authors survey our modern society and the nature of the secondary school program as a setting for the determination of what we should teach. Modern principles of learning are applied in illustrating the techniques, materials, and activities that have been found to be effective in teaching mathematics.

The authors begin appropriately by showing the role of mathematics in modern society. This concise chapter leaves the reader with an appreciation of the power of mathematics and a desire for more extensive information about the applications of mathematics. A review of dominant events in the development of secondary education furnishes a background for considering curriculum problems in mathematics. A chapter on how to direct learning in mathematics applies learning principles to the problem of teaching mathematics. Frequent illustrations of the techniques of specific master teachers of the subject are used to make the discussion realistic. In this way the book is unique in giving master teachers recognition for superior work.

A major portion of the book is given to a discussion of the aims, content, problems, and methods of teaching various fields of secondary and junior college mathematics. This discussion is made concrete by describing the current practices of good teachers. This manner of telling how to teach a topic is vivid and will be effective if the reader does not conclude that the technique described is the one and only method to be used. Although examples and materials are illustrated, no attempt is made to treat comprehensively all the topics of each area. It is the area of general mathematics which is receiving emphasis in the mathematics program of the secondary school today. Consequently the authors devote four chapters to the function, organization, administration and teaching of general mathematics from junior high school through junior college. Flow charts are used comparing the teacher's responsibilities with the order of pupil experiences in the development of concepts.

The remainder of the book deals with the use of teaching aids, planning units of instruction, constructing and using tests, and recreational mathematics. The chapter on teaching aids is devoted to methods of using teaching aids rather than a description of the great variety of materials which are available

to the teacher. The discussion of the long-unit assignment outlines the organization of a unit and illustrates them with parts of typical units. The description of tests includes a variety of types of test items to illustrate how to construct and use tests. It is unfortunate that this chapter is limited to achievement testing and, thus, does not treat the larger problem of evaluation which involves the measurement of objectives such as appreciations, attitudes, or work habits.

This book presents a wealth of practical hints for improving the teaching of mathematics. It is written in a brief, concise, readable form which is a distinct contrast to much literature in the field. However, the reviewer frequently had the feeling that the treatment of many topics was too brief. This briefness may be necessary in a text which treats a topic of such a wide scope as the teaching of mathematics from junior high school through junior college. Every teacher of mathematics will learn new ways of teaching old topics by reading this book.

DONOVAN A. JOHNSON
University of Minnesota

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB TOPICS

A recent pamphlet by C. B. Read, *Articles on the History of Mathematics*, contains an extensive bibliography which should prove very useful in planning club programs. The bibliography covers historical articles which have appeared in five periodicals prior to January 1, 1952: *The American Mathematical Monthly*, *The Mathematics Teacher*, *National Mathematics Magazine*, *Scripta Mathematica*, and *School Science and Mathematics*.

The material is divided into the following major categories, each of which suggests a possible club topic:

The history of mathematics as a whole
Mathematics in the ancient world
Mathematics in other countries
Mathematics in America
Numerals and number systems
Calculating methods and devices
Mathematical terms and symbols

Algebra and algebraic problems
Logarithms and exponentials
The numbers *e*, *pi*, and *i*
Complex numbers and the theory of numbers
Geometry and geometrical problems (analytic geometry, geometrical constructions, non-euclidean geometry, special geometrical curves, geometrical articles not otherwise specified)
Astronomy, trigonometry, surveying, and navigation
The calculus and related topics
Mathematical books and manuscripts
Mathematical periodicals
Mathematicians of ancient times
Mathematicians of the middle ages
Mathematicians of the sixteenth century
Mathematicians of the seventeenth century
Mathematicians of the eighteenth century
Mathematicians of the nineteenth and twentieth centuries
Miscellaneous.

The pamphlet was published as a Bulletin (University Studies No. 26) by the Municipal University of Wichita, Wichita, Kansas.

CLUB NEWS

The University of Virginia chapter of Pi Mu Epsilon in 1951–52 again sponsored prize examinations: one for students taking freshman mathematics, the other for students taking sophomore mathematics.

The Mathematics Club of the University of Massachusetts has awarded Allan Dickenson the fee for membership in the Mathematical Association of America for his outstanding work in the Mathematics Club during 1951–52.

During 1951–52, pledges of the Mount St. Scholastica College chapter of Kappa Mu Epsilon sponsored an open house, at which time the mathematics department received visitors for the entire college. Various mathematical concessions were arranged to amuse the guests. These consisted of optical illusions; brain teasers; identification of mathematical tools, figures, solids; and famous men of mathematics. Prizes were awarded to the most successful contestants.

PI MU EPSILON NATIONAL MEETING

A national meeting of Pi Mu Epsilon Fraternity was held at Michigan State College, East Lansing, Michigan, on Sunday evening, August 31, and Monday, September 1, 1952, in conjunction with meetings of the Mathematical Association of America, the American Mathematical Society, the Institute of Mathematical Statistics, and the Econometric Society.

On Sunday evening there was an informal discussion meeting followed by a meeting of the National Council. Members met for luncheon and business meeting in Phillips Hall.

The following papers were presented by Pi Mu Epsilon members on Monday:

So you want to be a graduate student, by Director General C. C. MacDuffee, Wisconsin Beta

Modification of infinite series, by Bevan K. Youse, Georgia Alpha

Rapid square roots, by Charles D. Parker, Michigan Alpha

Almost periodic functions, by John E. Hoffman, Oklahoma Alpha

Matrix inversion, by Verna Lair, Delaware Alpha

Metric extension, by Alan J. Goldman, New York Gamma

The G.C.D. algorithm, by Wilson M. Zaring, Kentucky Alpha.

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

Thirty-fourth Summer Meeting, Laval University, Quebec, Canada, August 31–September 1, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS

KENTUCKY, University of Louisville, Spring, 1953.

LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Howard University, Washington, D. C., December 6, 1952.

METROPOLITAN NEW YORK, Spring, 1953.

MICHIGAN, Western Michigan College of Education, Kalamazoo, April 11, 1953.

MINNESOTA, St. Olaf College, Northfield, May 9, 1953.

MISSOURI, William Jewell College, Liberty, Spring, 1953.

NEBRASKA

NORTHERN CALIFORNIA, San Francisco State College, January 31, 1953.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, Montana State University, Missoula, June, 1953.

PHILADELPHIA

ROCKY MOUNTAIN, University of Colorado, Boulder, April, 1953.

SOUTHEASTERN, Alabama Polytechnic Institute, Auburn, March 13–14, 1953.

SOUTHERN CALIFORNIA, Los Angeles City College, March 14, 1953.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanic Arts, State College, January 2–3, 1953.

TEXAS

UPPER NEW YORK STATE, United States Military Academy, West Point, Spring, 1953.

WISCONSIN, Mount Mary College, Milwaukee, May, 1953.

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